On the discrete spectrum of linear operators in Hilbert spaces

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Abstract

If $Z = Z_0 + M$ is a linear operator which arises from a closed operator Z_0 by some relatively compact perturbation M, then the essential spectra of Z and Z_0 coincide and the spectrum of Z can contain an at most countable sequence of isolated complex eigenvalues $\{\lambda_k\}$, which can accumulate on the essential spectrum only. The aim of this thesis is to provide estimates on the rate of accumulation of these eigenvalues, in terms of Schatten norm bounds on the operator $M(\lambda - Z_0)^{-1}$. More precisely, we will exploit the behavior of the $\mathcal{S}_p$-norm of $M(\lambda - Z_0)^{-1}$, for λ approaching the spectrum of Z_0, to derive estimates on $\sum_k \Phi_p(\lambda_k)$, where $\Phi_p : \mathbb{C} \to \mathbb{R}_+$ is a suitable continuous function which vanishes on the essential spectrum of Z. In particular, we will focus on the case that the operator Z_0 is selfadjoint and (semi)bounded.

We approach the problem of studying the isolated eigenvalues of Z by constructing a holomorphic function whose zeros coincide with the eigenvalues of Z and by using complex analysis to study these zeros.

Finally, the abstract results are applied to obtain Lieb-Thirring-type estimates on the eigenvalues of non-selfadjoint Jacobi and Schrödinger operators.

Zusammenfassung

Falls der lineare Operator $Z = Z_0 + M$ durch eine relativ kompakte Störung M aus dem abgeschlossenen Operator Z_0 hervorgeht, so stimmen die wesentlichen Spektren von Z und Z_0 überein und das Spektrum von Z kann eine höchstens abzählbare Folge von isolierten komplexen Eigenwerten $\{\lambda_k\}$ enthalten, welche sich nur beim wesentlichen Spektrum häufen können. Das Ziel dieser Arbeit ist es, Abschätzungen an die Häufungsrate dieser Eigenwerte in Abhängigkeit von Schatten-Norm Schranken an den Operator $M(\lambda - Z_0)^{-1}$ bereitzustellen. Genauer gesagt werden wir das Verhalten der $\mathcal{S}_p$-Norm von $M(\lambda - Z_0)^{-1}$, bei Annäherung von λ an das Spektrum von Z_0, ausnutzen, um Abschätzungen an $\sum_k \Phi_p(\lambda_k)$ zu erhalten, wobei $\Phi_p : \mathbb{C} \to \mathbb{R}_+$ eine geeignet gewählte stetige Funktion ist, die auf dem wesentlichen Spektrum von Z verschwindet. Unser Hauptaugenmerk liegt hierbei auf dem Fall, dass der Operator Z_0 selbstadjungiert und (halb)beschränkt ist.

Der von uns verwendete Ansatz zur Untersuchung der isolierten Eigenwerte von Z besteht in der Konstruktion einer holomorphen Funktion, deren Nullstellen mit den Eigenwerten von Z übereinstimmen, und in der Untersuchung dieser Nullstellen mit Mitteln der Funktionentheorie.

Die abstrakten Resultate werden schließlich angewandt, um Abschätzungen vom Lieb-Thirring-Typ für die Eigenwerte von nicht-selbstadjungierten Jacobi- und Schrödingeroperatoren zu gewinnen.

Contents

0. Introduction

Compared to the theory of selfadjoint operators, the corresponding non-selfadjoint theory is much more diverse. As E. Brian Davies put it in the preface of (Davies 2007),

> *"it can hardly be called a theory. (...) It comprises a collection of methods, each of which is useful for some class of such operators."*

– With the present thesis, we would like to contribute a new item to this collection.

The aim of this thesis is to use complex analysis to study the discrete spectrum—the set of isolated eigenvalues of finite multiplicity—of linear (non-selfadjoint) operators in Hilbert spaces, and to obtain estimates on these eigenvalues and on their rate of accumulation to the essential spectrum. To explain our goals in a little more detail, let us consider the following situation: Suppose that Z and Z_0 are closed linear operators and $Z = Z_0 + M$ where M is Z_0-compact. In this case, the essential spectra $\sigma_{ess}(Z)$ and $\sigma_{ess}(Z_0)$ coincide and the discrete spectrum $\sigma_d(Z)$ consists of an at most countable sequence of isolated eigenvalues which can accumulate to $\sigma_{ess}(Z_0)$ only. For simplicity, let us assume that $\sigma(Z_0) = \sigma_{ess}(Z_0)$ and $\sigma(Z) = \sigma(Z_0) \dot\cup \sigma_d(Z)$.

One way to obtain further information on the isolated eigenvalues of Z is to study the finiteness of $\sum_{\lambda \in \sigma_d(Z)} \Phi(\lambda)$, where $\Phi : \mathbb{C} \to \mathbb{R}_+$ is a suitable continuous function which vanishes on the spectrum of Z_0. For instance, we can choose $\Phi(\lambda) = \operatorname{dist}(\lambda, \sigma(Z_0))$ and ask for the existence of a constant $C(M)$ such that

$$\sum_{\lambda \in \sigma_d(Z)} \operatorname{dist}(\lambda, \sigma(Z_0)) \leq C(M). \tag{0.1}$$

Of course, the validity of this inequality will generally require some stronger assumptions on the perturbation M. For example, in the trivial case $Z_0 = 0$ inequality (0.1) is valid if M is a trace class operator and $C(M) = \|M\|_{\mathcal{S}_1}$ (here $\operatorname{dist}(\lambda, \sigma(Z_0)) = |\lambda|$). More generally, if $Z_0 = 0$ and M is in the Schatten class of order $p > 0$, then

$$\sum_{\lambda \in \sigma_d(Z)} \operatorname{dist}(\lambda, \sigma(Z_0))^p \leq \|M\|_{\mathcal{S}_p}^p. \tag{0.2}$$

Although the validity of the last inequality is far from being obvious if $Z_0 \neq 0$, and in this generality it will not be true at all, it certainly has some desirable features. Namely, if valid it would provide information on both the number of isolated eigenvalues of Z, showing that *outside* the set $\{\lambda : \operatorname{dist}(\lambda, \sigma(Z_0)) < \varepsilon\}$ there are no more than $\varepsilon^{-p}\|M\|_{\mathcal{S}_p}^p$ eigenvalues, and on the rate of accumulation of these eigenvalues to the spectrum of Z_0.

We will show in this thesis that estimates similar to (0.2) can indeed be derived by transferring the problem of studying the discrete spectrum of Z to a problem of analyzing the zero set of a holomorphic function. In a nutshell our method can be described as follows: Assuming that the operator $M(\lambda - Z_0)^{-1}$ is in the pth Schatten class, we use generalized determinants to construct a holomorphic function $d(\lambda)$ on $\mathbb{C} \setminus \sigma(Z_0)$ whose zero set coincides with the discrete spectrum of Z, and which satisfies an exponential bound of the form $\log |d(\lambda)| \leq C(p)\|M(\lambda - Z_0)^{-1}\|_{\mathbf{S}_p}^p$. Using conformal mappings to transfer the problem to the unit disk, we thus establish a correspondence between the discrete spectrum of Z and the zero set of a holomorphic function $h(w)$ on the unit disk, which explodes exponentially for w approaching the unit circle. Eventually, by establishing Blaschke-type estimates on the zeros of the function h, and by retranslating these estimates into estimates on $\sigma_d(Z)$, we obtain the desired analogs of (0.2).

The results presented in this thesis extend and unify several earlier results on the distribution of eigenvalues of non-selfadjoint operators that were obtained, using essentially the same approach as sketched above, by

- Demuth and Katriel, who developed the idea to use complex analysis to study the rate of accumulation of eigenvalues to the essential spectrum and applied it to obtain estimates on the discrete spectrum of selfadjoint Schrödinger operators (Demuth & Katriel 2008),

- Borichev, Golinskii and Kupin, who were the first to extend Demuth and Katriel's approach to the non-selfadjoint setting in order to study the eigenvalues of complex Jacobi operators (Borichev, Golinskii & Kupin 2009),

- Demuth, Hansmann and Katriel, who considered the discrete spectrum of general *unbounded* non-selfadjoint operators (Demuth, Hansmann & Katriel 2008), and who derived estimates on the eigenvalues of perturbations of non-negative operators to obtain Lieb-Thirring-type inequalities for Schrödinger operators with complex potentials (Demuth, Hansmann & Katriel 2009), and by

- Hansmann and Katriel, who modified the complex analysis result used by Borichev et al. and derived improved estimates on the discrete spectrum of complex Jacobi operators (Hansmann & Katriel 2009).

In the following, let us briefly summarize the contents of this thesis: In Chapter 1, we introduce various concepts from the spectral theory of linear operators. We start with a discussion of the relation between the essential and the discrete spectrum of non-selfadjoint operators, consider Weyl's theorem and its consequences on the essential spectrum, provide a short review of Schatten class operators and introduce the concept of infinite determinants on Hilbert spaces. The first chapter concludes with a detailed discussion of perturbation determinants and the relation of their zero sets to the discrete spectrum of the associated operators.

In Chapter 2, we study the distribution of zeros of holomorphic functions on the unit disk, growing exponentially near the boundary. Beginning with a short introduction why such functions naturally arise in the study of the discrete spectrum of linear operators, we continue with a presentation of various classical and recent results in this field. In the final section of this chapter we present a non-radial estimate due to Borichev, Golinskii and Kupin, which is particularly well-suited for our problems.

Chapter 3 can be regarded as the core of this thesis. We start with a presentation of some very general estimates on the discrete spectrum of Z in terms of estimates on the corresponding perturbation determinant, merely assuming that the resolvent difference $(a-Z)^{-1}-(a-Z_0)^{-1}$ is in $\mathcal{S}_p$, and continue with several more specialized estimates in the case where the operator Z_0 is selfadjoint.

In Chapter 4, the final chapter of the abstract part of this thesis, we show that some of the estimates derived in Chapter 3 can be improved if both Z and Z_0 are selfadjoint by using the variational characterization of the discrete spectrum. In particular, we will see that inequality (0.2) is valid in the selfadjoint case if we assume that the spectrum of Z_0 is an interval.

Finally, in Chapter 5 and 6 we apply our abstract results to derive Lieb-Thirring-type estimates on the discrete spectrum of non-selfadjoint Jacobi and Schrödinger operators.

Part I.

An abstract framework

1. Basic concepts and terminology

In this chapter we introduce and review some basic concepts of operator and spectral theory. As general references we refer to the monographs of Davies (2007), Gohberg, Goldberg & Kaashoek (1990) and Kato (1995).

1.1. The spectrum of linear operators

Summary: We introduce various concepts related to the spectrum of linear operators. In particular, we define the essential and the discrete spectrum of a linear operator and study their relation.

Let us begin with some notations. Throughout this thesis, $\mathcal{H}$ will denote a complex separable Hilbert space. If Z is a linear operator in $\mathcal{H}$ then we denote its **domain**, **range** and **kernel** by $\mathrm{Dom}(Z)$, $\mathrm{Ran}(Z)$ and $\mathrm{Ker}(Z)$, respectively (in the following, we will simply speak of operators in $\mathcal{H}$, taking their linearity for granted). We say that Z is an operator **on** $\mathcal{H}$ if $\mathrm{Dom}(Z) = \mathcal{H}$. The algebra of all **bounded operators** on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Similarly, $\mathcal{C}(\mathcal{H})$ denotes the class of all **closed operators** in $\mathcal{H}$. Furthermore, we denote the ideals of **compact** and **finite rank operators** on $\mathcal{H}$ by $\mathcal{S}_\infty(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$, respectively (the **rank** of Z is defined as $\mathrm{Rank}(Z) = \dim(\mathrm{Ran}(Z))$).

Let us agree that throughout this section Z denotes a closed operator in $\mathcal{H}$. We define the **resolvent set** of Z as

$$\rho(Z) = \{\lambda \in \mathbb{C} : \lambda - Z \text{ is invertible in } \mathcal{B}(\mathcal{H})\}.$$

Then $\rho(Z)$ is an open subset of the complex plane, and for $\lambda \in \rho(Z)$ we set

$$R_Z(\lambda) = (\lambda - Z)^{-1}. \tag{1.1}$$

The complement of $\rho(Z)$ in $\mathbb{C}$, denoted by $\sigma(Z)$, is called the **spectrum** of Z. In particular, we say that $\lambda \in \sigma(Z)$ is an **eigenvalue** of Z if $\mathrm{Ker}(\lambda - Z)$ is non-trivial.

If $Z \in \mathcal{B}(\mathcal{H})$ then $\lim_{\lambda\to\infty} R_Z(\lambda)$ exists and is equal to 0 (the zero-operator),[1] so in this case it is natural to consider $\lambda = \infty$ as an element of the resolvent set of Z. Consequently, for $Z \in \mathcal{B}(\mathcal{H})$ we set

$$R_Z(\infty) = 0. \tag{1.2}$$

[1] See, e.g., (Kato 1995), p.176.

Moreover, we introduce the **extended resolvent set** of Z as

$$\hat{\rho}(Z) = \begin{cases} \rho(Z) \cup \{\infty\}, & \text{if } Z \in \mathcal{B}(\mathcal{H}), \\ \rho(Z), & \text{if } Z \notin \mathcal{B}(\mathcal{H}). \end{cases} \tag{1.3}$$

In particular, if $Z \in \mathcal{B}(\mathcal{H})$ then $\hat{\rho}(Z)$ is an open subset of the **extended complex plane** $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (which is endowed with the usual topology).

Remark 1.1.1. If $\mathcal{X}$ is a Banach space and $\Omega \subset \hat{\mathbb{C}}$ is open and non-empty, then a function $g : \Omega \to \mathcal{X}$ is called *analytic* at $\lambda_0 \in \Omega \cap \mathbb{C}$ if in some neighborhood U of λ_0 in Ω it can be represented as

$$g(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n g_n, \quad \lambda \in U,$$

where $g_1, g_2, \ldots$ are vectors in $\mathcal{X}$ and the series converges in the norm of $\mathcal{X}$. If $\infty \in \Omega$ then g is called analytic at $\lambda_0 = \infty$ if the function $\mu \mapsto g(\mu^{-1})$ is analytic at $\mu_0 = 0$. Moreover, g is called analytic on Ω if it is analytic at each point of Ω.

Due to the fact that g is analytic on Ω if and only if it is *weakly analytic*, that is, the scalar valued function $l(g(.))$ is analytic on Ω for every continuous linear functional l on $\mathcal{X}$, the theory of vector-valued analytic functions is almost entirely analogous to the usual theory. In particular, g is analytic on Ω if and only if it is *holomorphic* on Ω, that is, if for every $\lambda_0 \in \Omega$ the derivative $g'(\lambda_0) = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{-1}(g(\lambda) - g(\lambda_0))$ exists in the norm of $\mathcal{X}$ (with $g'(\infty)$ being defined as $\lim_{\mu \to 0} \mu^{-1}(g(\mu^{-1}) - g(\infty))$). □

In view of the previous remark, we can now state the following proposition concerning the operator-valued function $\lambda \mapsto R_Z(\lambda)$, called the **resolvent** of Z.

Proposition 1.1.2. *The resolvent of Z is a $\mathcal{B}(\mathcal{H})$-valued analytic function on $\hat{\rho}(Z)$.*[2]

In other words, while $R_Z(.)$ is always analytic on $\rho(Z)$, it can be extended to an analytic function on $\hat{\rho}(Z)$ by setting $R_Z(\infty) = 0$ if Z is a bounded operator on $\mathcal{H}$.

We note that for every $\lambda \in \hat{\rho}(Z)$ the resolvent satisfies the norm inequality $\|R_Z(\lambda)\| \geq \operatorname{dist}(\lambda, \sigma(Z))^{-1}$, where $\|.\|$ denotes the norm of $\mathcal{B}(\mathcal{H})$,[3] and where we agree that $\frac{1}{\infty} = 0$. Actually, if Z is **selfadjoint**, i.e., $Z = Z^*$, then the spectral theorem implies the stronger identity

$$\|R_Z(\lambda)\| = \operatorname{dist}(\lambda, \sigma(Z))^{-1}, \quad \lambda \in \hat{\rho}(Z). \tag{1.4}$$

In the following, let us take a closer look at the spectrum of Z: If $\lambda \in \sigma(Z)$ is an isolated point of the spectrum then we define the **Riesz projection** of Z with respect to λ as

$$P_Z(\lambda) = \frac{1}{2\pi i} \int_\gamma R_Z(\mu) d\mu, \tag{1.5}$$

[2]See, e.g., (Kato 1995).

[3]We will use the same symbol to denote the norm of $\mathcal{H}$ as well.

where the contour γ is a counterclockwise oriented circle centered at λ, with sufficiently small radius (excluding the rest of $\sigma(Z)$).[4] In particular, $P_Z(\lambda) \in \mathcal{B}(\mathcal{H})$.

We recall that a subspace $\mathcal{W} \subset \mathcal{H}$ is called *Z-invariant* if $Z(\mathcal{W} \cap \mathrm{Dom}(Z)) \subset \mathcal{W}$. In this case, $Z|_{\mathcal{W}}$ denotes the *restriction* of Z to $\mathcal{W} \cap \mathrm{Dom}(Z)$, and the range of $Z|_{\mathcal{W}}$ is a subspace of $\mathcal{W}$.

Proposition 1.1.3. *Let $Z \in \mathcal{C}(\mathcal{H})$ and let $\lambda \in \sigma(Z)$ be isolated. If $P = P_Z(\lambda)$ is defined as above then the following holds:*

(i) *P is a projection, i.e., $P^2 = P$.*

(ii) $\mathrm{Ran}(P)$ *and* $\mathrm{Ker}(P)$ *are Z-invariant.*

(iii) $\mathrm{Ran}(P) \subset \mathrm{Dom}(Z)$ *and* $Z|_{\mathrm{Ran}(P)}$ *is bounded.*

(iv) $\sigma(Z|_{\mathrm{Ran}(P)}) = \{\lambda\}$ *and* $\sigma(Z|_{\mathrm{Ker}(P)}) = \sigma(Z) \setminus \{\lambda\}$.

For a proof we refer to (Gohberg et al. 1990), p.326.

We say that $\lambda_0 \in \sigma(Z)$ is an **eigenvalue of finite type** if λ_0 is an isolated point of $\sigma(Z)$ and $P = P_Z(\lambda_0)$ is of finite rank. Note that, in this case, λ_0 is indeed an eigenvalue of Z since $\{\lambda_0\} = \sigma(Z|_{\mathrm{Ran}(P)})$ and $\mathrm{Ran}(P)$ is Z-invariant and finite-dimensional. The positive integer

$$m_Z(\lambda_0) = \mathrm{Rank}(P_Z(\lambda_0)) \tag{1.6}$$

is called the **algebraic multiplicity** of λ_0 with respect to Z. It has to be distinguished from the **geometric multiplicity** of λ_0, which is defined as the dimension of $\mathrm{Ker}(\lambda_0 - Z)$.

Remark 1.1.4. If $\lambda_0 \in \sigma(Z)$ is an eigenvalue of finite type, then $\mathrm{Ran}(P_Z(\lambda_0))$ coincides with the *algebraic eigenspace* of Z, i.e., $\mathrm{Ran}(P_Z(\lambda_0)) = \cup_{n\in\mathbb{N}} \mathrm{Ker}((\lambda_0 - Z)^n)$ and there exists $n_0 \leq m_Z(\lambda_0)$ such that $\mathrm{Ker}((\lambda_0 - Z)^{n_0}) = \mathrm{Ker}((\lambda_0 - Z)^n)$ whenever $n \geq n_0$. In particular, the algebraic multiplicity of λ_0 is larger than or equal to the geometric multiplicity. □

Convention 1.1.5. In this thesis, only the algebraic multiplicity of eigenvalues is of importance. We will therefore use the term “multiplicity” as a synonym for “algebraic multiplicity”. □

By definition, the **discrete spectrum** of Z is the set of all of its eigenvalues of finite type. We denote it by $\sigma_d(Z)$, i.e.,

$$\sigma_d(Z) = \{\lambda \in \sigma(Z) : \lambda \text{ is an eigenvalue of finite type}\}. \tag{1.7}$$

To define the **essential spectrum**, we recall that a linear operator $Z_0 \in \mathcal{C}(\mathcal{H})$ is a **Fredholm operator** if it has closed range and both its kernel $\mathrm{Ker}(Z_0)$ and cokernel

[4] As in complex function theory, the integral in (1.5) is defined as a Stieltjes integral, with the only difference to the scalar case that its convergence is to be understood in the norm of $\mathcal{B}(\mathcal{H})$.

$\mathcal{H}/\operatorname{Ran}(Z_0)$ are finite-dimensional. Here $\mathcal{H}/\operatorname{Ran}(Z_0)$ denotes the quotient space of $\operatorname{Ran}(Z_0)$ with respect to $\mathcal{H}$. Equivalently, if $Z_0 \in \mathcal{C}(\mathcal{H})$ is densely defined then Z_0 is Fredholm if and only if it has closed range and both $\operatorname{Ker}(Z_0)$ and $\operatorname{Ker}(Z_0^*) = \operatorname{Ran}(Z_0)^\perp$ are finite-dimensional. The essential spectrum of Z can now be defined as follows:

$$\sigma_{ess}(Z) = \{\lambda \in \mathbb{C} : \lambda - Z \text{ is not a Fredholm operator}\}.^5 \tag{1.8}$$

Note that $\sigma_{ess}(Z) \subset \sigma(Z)$ and that $\sigma_{ess}(Z)$ is a closed set.[6] We will see in the next section that, in contrast to the discrete spectrum, the essential spectrum is stable under (relatively) compact perturbations of the operator.

Before continuing with some remarks on the relation between $\sigma_d(Z)$ and $\sigma_{ess}(Z)$, let us consider the spectrum of the resolvent of Z.

Proposition 1.1.6. *Let $Z \in \mathcal{C}(\mathcal{H})$ with $\rho(Z) \neq \emptyset$. If $a \in \rho(Z)$ then*

$$\sigma(R_Z(a)) \setminus \{0\} = \{(a-\lambda)^{-1} : \lambda \in \sigma(Z)\}.$$

The same identity holds when, on both sides, σ is replaced by σ_{ess} and σ_d, respectively. More precisely, λ_0 is an isolated point of $\sigma(Z)$ if and only if $(a-\lambda_0)^{-1}$ is an isolated point of $\sigma(R_Z(a))$, and in this case

$$P_Z(\lambda_0) = P_{R_Z(a)}((a-\lambda_0)^{-1}).$$

In particular, the algebraic multiplicities of $\lambda_0 \in \sigma_d(Z)$ and $(a-\lambda_0)^{-1} \in \sigma_d(R_Z(a))$ coincide.

For a proof we refer to (Engel & Nagel 2000), p.243 and p.247, and to (Davies 2007), p.331.

Remark 1.1.7. We note that $0 \in \sigma(R_Z(a))$ if and only if $Z \notin \mathcal{B}(\mathcal{H})$. Moreover, if $Z \in \mathcal{C}(\mathcal{H})$ is densely defined, then

$$0 \in \sigma(R_Z(a)) \quad \Leftrightarrow \quad 0 \in \sigma_{ess}(R_Z(a)).$$

$\square$

The following proposition shows that the essential and the discrete spectrum of a linear operator are disjoint.

Proposition 1.1.8. *Let $Z \in \mathcal{C}(\mathcal{H})$ and let $\lambda \in \sigma(Z)$ be isolated. Then $\lambda \in \sigma_{ess}(Z)$ if and only if $\operatorname{Rank}(P_Z(\lambda)) = \infty$. In particular,*

$$\sigma_{ess}(Z) \cap \sigma_d(Z) = \emptyset.$$

[5]For a discussion of various alternative (non-equivalent) definitions of the essential spectrum see (Edmunds & Evans 1987). We note that all reasonable definitions coincide in the selfadjoint case.

[6]For instance, this is a consequence of the fact that $Z_0 + M$ is Fredholm if Z_0 is Fredholm and $\|M\|$ is sufficiently small, see, e.g., (Gohberg et al. 1990), p.189.

Proof. For $Z \in \mathcal{B}(\mathcal{H})$ a proof can be found in (Davies 2007), p.122. The unbounded case can be reduced to the bounded case by means of Proposition 1.1.6. ■

While the spectrum of a selfadjoint operator Z can always be decomposed as

$$\sigma(Z) = \sigma_{ess}(Z) \,\dot{\cup}\, \sigma_d(Z), \tag{1.9}$$

where the symbol $\dot{\cup}$ denotes a disjoint union, the same decomposition need not be true in the non-selfadjoint case. For instance, considering the shift operator $(Zf)(n) = f(n+1)$ acting on $l^2(\mathbb{N})$, we have $\sigma_{ess}(Z) = \{z \in \mathbb{C} : |z| = 1\}$ and $\sigma(Z) = \{z \in \mathbb{C} : |z| \leq 1\}$, while $\sigma_d(Z) = \emptyset$. For a proof see (Kato 1995), p.237-238.

The spectrum of the shift operator also nicely illustrates the contents of the following proposition.

Proposition 1.1.9. *Let $Z \in \mathcal{C}(\mathcal{H})$ and let $\Omega \subset \mathbb{C} \setminus \sigma_{ess}(Z)$ be open and connected. If $\Omega \cap \rho(Z) \neq \emptyset$ then $\sigma(Z) \cap \Omega \subset \sigma_d(Z)$.*

A proof can be found in (Gohberg et al. 1990), p.373.

Hence, if Ω is a maximal connected component of $\mathbb{C} \setminus \sigma_{ess}(Z)$ then either

(i) $\Omega \subset \sigma(Z)$ (in particular, $\Omega \cap \sigma_d(Z) = \emptyset$), or

(ii) $\Omega \cap \rho(Z) \neq \emptyset$ and $\Omega \cap \sigma(Z)$ consists of an at most countable sequence of eigenvalues of finite type which can accumulate at $\sigma_{ess}(Z)$ only.

We conclude this section with a simple sufficient condition for the validity of (1.9) in the non-selfadjoint case. It is a direct consequence of Proposition 1.1.9.

Corollary 1.1.10. *Let $Z \in \mathcal{C}(\mathcal{H})$ with $\sigma_{ess}(Z) \subset \mathbb{R}$ and assume that there are points of $\rho(Z)$ in both the upper and lower half-planes. Then $\sigma(Z) = \sigma_{ess}(Z) \,\dot{\cup}\, \sigma_d(Z)$.*

Remark 1.1.11. If $\sigma_{ess}(Z) \subsetneq \mathbb{R}$ and $\rho(Z) \neq \emptyset$, then Proposition 1.1.9 already implies that there are points of $\rho(Z)$ in *both* the upper and lower half-planes. □

Of course, Proposition 1.1.9 would allow for a more general formulation of the last corollary. However, for our purposes the above formulation is general enough.

1.2. Weyl's theorem and its consequences

Summary: We discuss Weyl's theorem on the invariance of the essential spectrum under relatively compact perturbations.

Let us begin this section by introducing the notions of relative boundedness and relative compactness. To this end, in the following, let Z_0 denote a closed operator in $\mathcal{H}$ with non-empty resolvent set. We say that the operator M in $\mathcal{H}$ is Z_0-**bounded** (or relatively bounded with respect to Z_0) if $\mathrm{Dom}(Z_0) \subset \mathrm{Dom}(M)$ and there exist two non-negative numbers r and s such that

$$\|Mf\| \leq r\|f\| + s\|Z_0 f\|$$

for all $f \in \mathrm{Dom}(Z_0)$. The infimum of all constants s for which a corresponding r exists such that the last inequality holds is called the Z_0-**bound** of M. It is not difficult to see that M is Z_0-bounded if and only if $\mathrm{Dom}(Z_0) \subset \mathrm{Dom}(M)$ and $MR_{Z_0}(a) \in \mathcal{B}(\mathcal{H})$ for some $a \in \rho(Z_0)$. Moreover, the Z_0-bound of M is not larger than $\inf_{a\in\rho(Z_0)} \|MR_{Z_0}(a)\|$.

Remark 1.2.1. If $MR_{Z_0}(a) \in \mathcal{B}(\mathcal{H})$ for some $a \in \rho(Z_0)$, then $MR_{Z_0}(b) \in \mathcal{B}(\mathcal{H})$ for every $b \in \rho(Z_0)$. This is a consequence of the **first resolvent identity**

$$R_{Z_0}(a) - R_{Z_0}(b) = (b - a)R_{Z_0}(a)R_{Z_0}(b). \tag{1.10}$$

□

The operator M is called Z_0-**compact** (or relatively compact with respect to Z_0) if $\mathrm{Dom}(Z_0) \subset \mathrm{Dom}(M)$ and for any sequence $f_n \in \mathrm{Dom}(Z_0)$ with both $\{f_n\}$ and $\{Z_0 f_n\}$ bounded, $\{Mf_n\}$ contains a convergent subsequence. Alternatively, we have the following characterization of Z_0-compactness.

Proposition 1.2.2. *M is Z_0-compact if and only if* $\mathrm{Dom}(Z_0) \subset \mathrm{Dom}(M)$ *and $MR_{Z_0}(a) \in \mathcal{S}_\infty(\mathcal{H})$ for some (and hence all) $a \in \rho(Z_0)$.*

The last characterization shows that every Z_0-compact operator is Z_0-bounded and it can also be shown that in this case the corresponding Z_0-bound is 0. Moreover, if $Z_0 \in \mathcal{B}(\mathcal{H})$ then every Z_0-compact operator is also compact (the inverse implication being true in any case).

In the following, let us consider an operator Z in $\mathcal{H}$ which arises from Z_0 by some relatively bounded perturbation. In other words, Z can be defined as $Z = Z_0 + M$[7] where M is Z_0-bounded. The aim of perturbation theory is to obtain spectral and other properties of Z, assuming that the operator Z_0 is well-known and can be analyzed in great detail. For instance, a standard result of perturbation theory tells us that Z is closed together with Z_0 if the Z_0-bound of M is smaller than 1, thus reducing the problem of determining whether Z is closed to a study of the operator $MR_{Z_0}(a)$.

In the remaining part of this section, we will provide similar perturbation results related to the essential spectrum of Z.

Theorem 1.2.3. *Let $Z_0 \in \mathcal{C}(\mathcal{H})$ and let M be Z_0-compact. If Z_0 is Fredholm, then $Z_0 + M$ is Fredholm as well.*

A proof can be found in (Kato 1995), p.238.

[7] Throughout this thesis, the sum $Z_1 + Z_2$ of two operators Z_1 and Z_2 in $\mathcal{H}$ denotes the usual operator sum defined on $\mathrm{Dom}(Z_1) \cap \mathrm{Dom}(Z_2)$.

We will refer to the next theorem as **Weyl's theorem**.

Theorem 1.2.4. *Let $Z = Z_0 + M$ where $Z_0 \in \mathcal{C}(\mathcal{H})$ and M is Z_0-compact. Then $\sigma_{ess}(Z) = \sigma_{ess}(Z_0)$.*

Proof. We begin with a general remark: If $A, B \in \mathcal{C}(\mathcal{H})$ and B is A-compact, then B is $(B + A)$-compact as well, see (Kato 1995), p.194. Furthermore, B is $(\lambda - A)$-compact for every $\lambda \in \mathbb{C}$, which is a direct consequence of the definition of relative compactness. Using this remark, the equivalence

$$\lambda - Z_0 \text{ is Fredholm} \quad \Leftrightarrow \quad \lambda - Z_0 - M \text{ is Fredholm}$$

is an immediate consequence of Theorem 1.2.3. ■

Remark 1.2.5. Whereas Weyl's theorem shows that the essential spectrum of linear operators is stable under relatively compact perturbations, a similar statement for the discrete spectrum is far from being true. For instance, we can construct a linear operator Z_0 with no eigenvalues and a rank one operator M such that Z_0+M has infinitely many eigenvalues of finite type, see Appendix C. □

The following proposition is a consequence of Weyl's theorem and Proposition 1.1.6. It provides a sufficient condition for the equality of the essential spectra of two operators Z_1 and Z_2, which is applicable even in case their difference $Z_2 - Z_1$ can not be suitably defined.

Proposition 1.2.6. *Let $Z_1, Z_2 \in \mathcal{C}(\mathcal{H})$ with $\rho(Z_1) \cap \rho(Z_2) \neq \emptyset$. If $R_{Z_2}(a) - R_{Z_1}(a) \in \mathcal{S}_\infty(\mathcal{H})$ for some $a \in \rho(Z_1) \cap \rho(Z_2)$ then $\sigma_{ess}(Z_2) = \sigma_{ess}(Z_1)$.*

Remark 1.2.7. Once again, if $R_{Z_2}(a) - R_{Z_1}(a) \in \mathcal{S}_\infty(\mathcal{H})$ for some $a \in \rho(Z_1) \cap \rho(Z_2)$, then the same is true for every $a \in \rho(Z_1) \cap \rho(Z_2)$. This is a consequence of the identity

$$R_{Z_2}(b) - R_{Z_1}(b) = (a - Z_2)R_{Z_2}(b)(R_{Z_2}(a) - R_{Z_1}(a))(a - Z_1)R_{Z_1}(b),$$

valid for $a, b \in \rho(Z_1) \cap \rho(Z_2)$. □

Combining the last proposition with Corollary 1.1.10 we obtain the following result.

Proposition 1.2.8. *Let $Z_1, Z_2 \in \mathcal{C}(\mathcal{H})$ where Z_1 is selfadjoint. In addition, suppose that there are points of $\rho(Z_2)$ in both the upper and lower half-planes. If $R_{Z_2}(a) - R_{Z_1}(a) \in \mathcal{S}_\infty(\mathcal{H})$ for some $a \in \rho(Z_1) \cap \rho(Z_2)$, then $\sigma_{ess}(Z_2) = \sigma_{ess}(Z_1) \subset \mathbb{R}$ and*

$$\sigma(Z_2) = \sigma_{ess}(Z_1) \,\dot{\cup}\, \sigma_d(Z_2). \tag{1.11}$$

Remark 1.2.9. If $Z = Z_0 + M$ where Z_0 is selfadjoint and M is Z_0-compact, then the intersection of $\rho(Z)$ with both the upper and lower half-plane is non-empty (for instance, this is a consequence of Theorem 11.1.3 in (Davies 2007), p.326). Moreover, if $a \in \rho(Z) \cap \rho(Z_0)$ then $R_Z(a) - R_{Z_0}(a) \in \mathcal{S}_\infty(\mathcal{H})$ as a consequence of the **second resolvent identity**

$$R_Z(a) - R_{Z_0}(a) = R_Z(a) M R_{Z_0}(a). \tag{1.12}$$

Hence, Proposition 1.2.8 implies that $\sigma_{ess}(Z) = \sigma_{ess}(Z_0)$ and $\sigma(Z) = \sigma_{ess}(Z_0) \,\dot{\cup}\, \sigma_d(Z)$. Alternatively, this can also be shown directly using Weyl's theorem. □

1.3. Schatten class operators

Summary: We introduce Schatten ideals to measure the degree of compactness of linear operators. In particular, we study the eigenvalues and singular values of Schatten class operators and show that their respective distributions are closely related.

We refer to (Gohberg, Goldberg & Krupnik 2000) as an additional reference for this section.

Let $B \in \mathcal{B}(\mathcal{H})$. For $n \in \mathbb{N}$ we define the nth **singular value** of B by setting

$$s_n(B) = \inf\{\|B - F\| : F \in \mathcal{F}(\mathcal{H}),\ \mathrm{Rank}(F) \leq n-1\}. \tag{1.13}$$

Then $s_1(B) = \|B\|$ and the sequence $\{s_n(B)\}$ is non-increasing. Moreover, the operator B is compact if and only if $\lim_{n\to\infty} s_n(B) = 0$. In the following, we will obtain a finer classification of compact operators by imposing summability properties on their singular values.

Let us begin with a definition: The class of all compact operators on $\mathcal{H}$ whose singular values are p-summable is called the **Schatten class**[8] of order p (here $p \in (0,\infty)$). We denote it by $\mathcal{S}_p(\mathcal{H})$. In other words,

$$K \in \mathcal{S}_p(\mathcal{H}) \quad \Leftrightarrow \quad \{s_n(K)\} \in l^p(\mathbb{N}). \tag{1.14}$$

We remark that $\mathcal{S}_p(\mathcal{H})$ is a subspace of $\mathcal{S}_\infty(\mathcal{H})$ for every $p > 0$. Moreover, if $p \geq 1$ then it becomes a complete normed space by introducing the norm

$$\|K\|_{\mathcal{S}_p} = \|\{s_n(K)\}\|_{l^p}. \tag{1.15}$$

Note that we will use the last definition in case that $p < 1$ (where it only defines a quasi-norm) as well. For consistency, we set $\|K\|_{\mathcal{S}_\infty} = \|K\|$.

The following inclusion, which readily follows from the definition, shows that Schatten classes can indeed be used to classify compact operators: if $0 < p \leq q \leq \infty$ then

$$\mathcal{S}_p(\mathcal{H}) \subset \mathcal{S}_q(\mathcal{H}). \tag{1.16}$$

Clearly, if $p < q$ then this inclusion is strict. More precisely, if $K \in \mathcal{S}_p(\mathcal{H})$ and $p \leq q$, then

$$\|K\|_{\mathcal{S}_q} \leq \|K\|_{\mathcal{S}_p}. \tag{1.17}$$

Similar to the class of compact operators, $\mathcal{S}_p(\mathcal{H})$ is a two-sided ideal in the algebra $\mathcal{B}(\mathcal{H})$, i.e., if $K \in \mathcal{S}_p(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ then $KB \in \mathcal{S}_p(\mathcal{H})$ and $BK \in \mathcal{S}_p(\mathcal{H})$ with

$$\|KB\|_{\mathcal{S}_p} \leq \|K\|_{\mathcal{S}_p}\|B\| \quad \text{and} \quad \|BK\|_{\mathcal{S}_p} \leq \|B\|\|K\|_{\mathcal{S}_p}. \tag{1.18}$$

Moreover, if $K \in \mathcal{S}_p(\mathcal{H})$ then $K^* \in \mathcal{S}_p(\mathcal{H})$ and $\|K^*\|_{\mathcal{S}_p} = \|K\|_{\mathcal{S}_p}$ (being a consequence of the fact that K and K^* have the same singular values).

[8] sometimes also referred to as *von Neumann-Schatten class.*

The next proposition provides a Schatten norm version of **Hölder's inequality**. A proof can be found in (Gohberg et al. 2000), p.88.

Proposition 1.3.1. *Let $K_1 \in \mathcal{S}_p(\mathcal{H})$ and $K_2 \in \mathcal{S}_q(\mathcal{H})$ where $0 < p, q \leq \infty$. Then $K_1K_2 \in \mathcal{S}_r(\mathcal{H})$, where $r^{-1} = p^{-1} + q^{-1}$, and*

$$\|K_1K_2\|_{\mathcal{S}_r} \leq \|K_1\|_{\mathcal{S}_p}\|K_2\|_{\mathcal{S}_q}.$$

Let us recall that the spectrum of a compact operator K on $\mathcal{H}$ consists of an at most countable sequence of eigenvalues of finite type, which can accumulate at 0 only[9] (here the point 0 may or may not belong to the spectrum). In other words, we have

$$\sigma(K) \setminus \{0\} = \sigma_d(K) \setminus \{0\}. \tag{1.19}$$

More precisely, if $\mathcal{H}$ is infinite-dimensional then $\sigma_{ess}(K) = \{0\}$, since the essential spectrum of a bounded operator on an infinite-dimensional Hilbert space is always non-empty (this is a consequence of Proposition 1.1.9).

Remark 1.3.2. Let K be a compact operator on $\mathcal{H}$ and let

$$\lambda_1(K^*K) \geq \lambda_2(K^*K) \geq \dots \tag{1.20}$$

denote the sequence of non-zero eigenvalues of the non-negative compact operator K^*K (counted according to multiplicity). The number of non-zero eigenvalues of K^*K is finite if and only if K is of finite rank, and in that case we extend the sequence (1.20) by zero elements so that in all cases (1.20) is an infinite sequence. Then for all $n \in \mathbb{N}$ we have $s_n(K) = \sqrt{\lambda_n(K^*K)}$. □

In view of their definition, it is reasonable to assume that for Schatten class operators, apart from (1.19), we should obtain some additional information on the discrete spectrum. Indeed, if $K \in \mathcal{S}_p(\mathcal{H})$ is selfadjoint and non-negative, then the previous remark implies that its eigenvalues and singular values coincide, so the eigenvalues of K are p-summable. While the eigenvalues and singular values of general compact operators need not coincide, the following result due to Weyl shows that their summability properties remain closely related.

Proposition 1.3.3. *Let $K \in \mathcal{S}_p(\mathcal{H})$, where $0 < p < \infty$, and let $\lambda_1, \lambda_2, \dots$ denote its sequence of non-zero eigenvalues (ordered according to decreasing absolute values, counted according to multiplicity and extended by zero elements to an infinite sequence if necessary). Then for $N \in \mathbb{N} \cup \{\infty\}$ we have*

$$\sum_{n=1}^{N} |\lambda_n|^p \leq \sum_{n=1}^{N} s_n(K)^p. \tag{1.21}$$

[9] For instance, this follows from Remark 1.2.9 by regarding K as a compact perturbation of the zero operator.

A proof can be found in (Gohberg et al. 2000), p.54.

Remark 1.3.4. We emphasize that, in the non-selfadjoint case, there is no individual estimate of the form $|\lambda_n| \leq s_n(K)$. □

In particular, Proposition 1.3.3 shows that for $K \in \mathcal{S}_1(\mathcal{H})$ the **trace** $\operatorname{tr}(K) = \sum_n \lambda_n$ is well defined and satisfies

$$|\operatorname{tr}(K)| \leq \|K\|_{\mathcal{S}_1}. \tag{1.22}$$

We note that $K \mapsto \operatorname{tr}(K)$ is a continuous *linear* functional on $\mathcal{S}_1(\mathcal{H})$ and that $\mathcal{S}_1(\mathcal{H})$ is also referred to as the *trace class.*

Let us conclude this section with a remark on the singular values of AB and BA where A and B are bounded operators on $\mathcal{H}$. While these two products have the same non-zero eigenvalues (with coinciding multiplicities),[10] they don't need to have the same singular values. For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $s_1(AB) = s_2(AB) = 0$ but $s_1(BA) = 1$ and $s_2(BA) = 0$. The next proposition shows that the singular values of AB and BA do coincide in the selfadjoint case.

Proposition 1.3.5. *If $A, B \in \mathcal{B}(\mathcal{H})$ are selfadjoint then $s_n(AB) = s_n(BA)$. In particular, $AB \in \mathcal{S}_p(\mathcal{H})$ if and only if $BA \in \mathcal{S}_p(\mathcal{H})$, and $\|AB\|_{\mathcal{S}_p} = \|BA\|_{\mathcal{S}_p}$.*

Proof. We have $s_n(AB) = s_n((AB)^*) = s_n(B^*A^*) = s_n(BA)$. ■

1.4. Determinants on Hilbert spaces

Summary: Generalizing the finite-dimensional case, we introduce (regularized) determinants for operators on general Hilbert spaces and study their properties.

For a more thorough discussion of the topics in this section we refer to (Simon 2005), Chapters 3 and 9.

Let $K \in \mathcal{S}_n(\mathcal{H})$, where $n \in \mathbb{N}$, and let $\lambda_1, \lambda_2, \ldots$ (where $|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$) denote its sequence of non-zero eigenvalues, counted according to multiplicity and extended by zero elements to an infinite sequence if necessary. We define the n-**regularized determinant** of $I - K$, where I denotes the identity operator on $\mathcal{H}$, as follows:

$$\det_n(I-K) = \begin{cases} \prod_{k\in\mathbb{N}}(1-\lambda_k), & \text{if } n = 1, \\ \prod_{k\in\mathbb{N}}\left[(1-\lambda_k)\exp\left(\sum_{j=1}^{n-1}\frac{\lambda_k^j}{j}\right)\right], & \text{if } n \geq 2. \end{cases} \tag{1.23}$$

Note that the right-hand side of (1.23) is a canonical product of genus $n-1$ associated with the sequence $\{\lambda_k^{-1}\}$, see, e.g., (Ahlfors 1978), p.196. Its convergence is a consequence of estimate (1.21).

[10]See, e.g., (König 1986), p.25.

It follows from the definition that $I - K$ is invertible in $\mathcal{B}(\mathcal{H})$ if and only if $\det_n(I - K) \neq 0$. Moreover, we have $\det_n(I) = 1$. In the following, we state some further properties of the regularized determinant.

Proposition 1.4.1. *If A and B are bounded operators on $\mathcal{H}$, with $AB, BA \in \mathcal{S}_n(\mathcal{H})$, then*

$$\det_n(I - AB) = \det_n(I - BA). \tag{1.24}$$

Proof. This is a consequence of the fact that the non-zero eigenvalues of AB and BA coincide. ■

The regularized determinant $\det_n(I-K)$ is a continuous function of K, see, e.g., (Simon 2005), p.75. The following result is even more important.

Proposition 1.4.2. *If $\Omega \subset \hat{\mathbb{C}}$ is open and $\lambda \mapsto K(\lambda)$ is a $\mathcal{S}_n(\mathcal{H})$-valued holomorphic function[11] on Ω, then $\lambda \mapsto \det_n(I - K(\lambda))$ is holomorphic on Ω as well.*

For a proof we refer to (Simon 1977).

As we will see in the next section, the previous proposition is one of the main tools in our construction of a holomorphic function whose zero set coincides with the discrete spectrum of a given linear operator Z. Indeed, Proposition 1.4.2 and the above considerations reduce this problem to a construction of a suitable $\mathcal{S}_n$-valued holomorphic function $K(\lambda)$, which satisfies the equivalence

$$\lambda \in \sigma_d(Z) \quad \Leftrightarrow \quad I - K(\lambda) \text{ is not invertible.}$$

In this context, the restriction to integer-valued Schatten classes is often too restrictive, i.e., we will have to consider $K \in \mathcal{S}_p(\mathcal{H})$ for some arbitrary $p \in (0, \infty)$ as well. In this case, however, since $\mathcal{S}_p(\mathcal{H}) \subset \mathcal{S}_{\lceil p \rceil}(\mathcal{H})$ where

$$\lceil p \rceil = \min\{n \in \mathbb{N} : n \geq p\}, \tag{1.25}$$

the $\lceil p \rceil$-regularized determinant of $I - K$ is well defined, so the above results can still be applied. Moreover, the following proposition (which can be found in (Dunford & Schwartz 1963), p.1106) provides us with an estimate on this determinant in terms of the pth Schatten norm of K.

Proposition 1.4.3. *Let $K \in \mathcal{S}_p(\mathcal{H})$ where $0 < p < \infty$. Then*

$$|\det_{\lceil p \rceil}(I - K)| \leq \exp\left(\Gamma_p \|K\|_{\mathcal{S}_p}^p\right), \tag{1.26}$$

where Γ_p is some positive constant.

We note that $\Gamma_p = \frac{1}{p}$ if $p \leq 1$, $\Gamma_2 = \frac{1}{2}$ and $\Gamma_n \leq e(2 + \log n)$ if n is an integer not less than 3, see (Simon 1977). A more recent study on the size of the constants Γ_n, with $n \in \mathbb{N}$, can be found in (Gil' 2008).

[11] See Remark 1.1.1.

1.5. Perturbation determinants

Summary: We use the regularized determinants defined in the previous section to transfer the problem of analyzing the discrete spectrum of a linear operator to an analysis of the zero set of a holomorphic function.

Throughout this section we make the following assumption.

Assumption 1.5.1. Z_0 and Z are operators in $\mathcal{H}$ such that

(i) Z_0, Z are closed and densely defined with $\rho(Z_0) \cap \rho(Z) \neq \emptyset$,

(ii) $R_Z(b) - R_{Z_0}(b) \in \mathcal{S}_p(\mathcal{H})$ for some $b \in \rho(Z_0) \cap \rho(Z)$ and some fixed $p \in (0, \infty)$,

(iii) $\sigma_d(Z) = \sigma(Z) \cap \rho(Z_0)$.

Remark 1.5.2. By Proposition 1.2.8, assumption (iii) follows from assumption (ii) if Z_0 is selfadjoint with $\sigma_d(Z_0) = \emptyset$ and there exist points of $\rho(Z)$ in both the upper and lower half-planes. □

Remark 1.5.3. The methods described below still work when assumption (iii) is replaced with the more general assumption that $\sigma_d(Z) \cap \rho(Z_0) = \sigma(Z) \cap \rho(Z_0)$. However, since it allows for a slightly easier presentation, and in view of later applications, we think that this loss of generality is justified. □

Of course, we do not exclude the possibility that Z_0 and Z are bounded operators on $\mathcal{H}$. In this case, the second resolvent identity implies that assumption (ii) is equivalent to the assumption that $Z - Z_0 \in \mathcal{S}_p(\mathcal{H})$.

In the following, we will construct an analytic function whose zero set coincides with the discrete spectrum of Z. To begin, let us construct such a function for the special case when $Z_0, Z \in \mathcal{B}(\mathcal{H})$: In this case, for $\lambda_0 \in \rho(Z_0)$ the identity $(\lambda_0 - Z)R_{Z_0}(\lambda_0) = I - (Z - Z_0)R_{Z_0}(\lambda_0)$ shows that $\lambda_0 \in \sigma_d(Z)$ if and only if $I - (Z - Z_0)R_{Z_0}(\lambda_0)$ is not invertible (recall that $\sigma(Z) \cap \rho(Z_0) = \sigma_d(Z)$ by assumption). But we already know from the previous section that this operator is not invertible if and only if

$$\det_{\lceil p \rceil}(I - (Z - Z_0)R_{Z_0}(\lambda_0)) = 0.$$[12]

Therefore, since $\lambda \mapsto (Z - Z_0)R_{Z_0}(\lambda)$ is an $\mathcal{S}_{\lceil p \rceil}$-valued analytic function on $\hat{\rho}(Z_0)$, Proposition 1.4.2 implies that $\lambda_0 \in \sigma_d(Z)$ if and only if λ_0 is a zero of the analytic function

$$d_\infty^{Z,Z_0} : \hat{\rho}(Z_0) \to \mathbb{C}, \qquad d_\infty^{Z,Z_0}(\lambda) = \det_{\lceil p \rceil}(I - (Z - Z_0)R_{Z_0}(\lambda)). \tag{1.27}$$

Our task is completed by noting that, as we will show below, the order of λ_0 as a zero of d_∞^{Z,Z_0} coincides with its multiplicity as an eigenvalue of Z. For later purposes, we also note that $d_\infty^{Z,Z_0}(\infty) = 1$, which should also explain the meaning of the index "∞".

Next, we consider the general case when Z_0 and Z merely fulfill Assumption 1.5.1 (of course, the operators may still be bounded; however, this is not an assumption anymore).

[12] Note that $(Z - Z_0)R_{Z_0}(\lambda) \in \mathcal{S}_p(\mathcal{H})$ since $(Z - Z_0) \in \mathcal{S}_p(\mathcal{H})$ by assumption.

Remark 1.5.4. Let $a \in \rho(Z_0) \cap \rho(Z)$ where Z_0, Z satisfy Assumption 1.5.1. Then Proposition 1.1.6 and its accompanying remark show that

$$\sigma_d(R_Z(a)) = \sigma(R_Z(a)) \cap \rho(R_{Z_0}(a)).$$

Hence, $\tilde{Z}_0 = R_{Z_0}(a)$ and $\tilde{Z} = R_Z(a)$ satisfy Assumption 1.5.1 as well. $\square$

In the following, let $a \in \rho(Z_0) \cap \rho(Z)$ be fixed. The last remark and our previous considerations show that the function

$$d_\infty^{R_Z(a),R_{Z_0}(a)}(.) = \det_{\lceil p \rceil}(I - [R_Z(a) - R_{Z_0}(a)][(.) - R_{Z_0}(a)]^{-1}) \tag{1.28}$$

is well defined and analytic on $\hat{\rho}(R_{Z_0}(a))$. Since $\lambda \in \hat{\rho}(Z_0)$ if and only if $(a-\lambda)^{-1} \in \hat{\rho}(R_{Z_0}(a))$ (which is again a consequence of Proposition 1.1.6 and Remark 1.1.7), we thus see that the function

$$d_a^{Z,Z_0}(\lambda) = d_\infty^{R_Z(a),R_{Z_0}(a)}((a-\lambda)^{-1}) \tag{1.29}$$

is analytic on $\hat{\rho}(Z_0)$[13] and satisfies

$$d_a^{Z,Z_0}(\lambda) = 0 \quad \Leftrightarrow \quad (a-\lambda)^{-1} \in \sigma_d(R_Z(a)) \quad \Leftrightarrow \quad \lambda \in \sigma_d(Z)$$

as desired. Moreover, as above we have $d_a^{Z,Z_0}(a) = d_\infty^{R_Z(a),R_{Z_0}(a)}(\infty) = 1$.

Let us summarize the previous discussion with the following proposition.

Proposition 1.5.5. *Let $a \in \hat{\rho}(Z_0) \cap \hat{\rho}(Z)$ where Z, Z_0 satisfy Assumption 1.5.1, and let $d_a = d_a^{Z,Z_0} : \hat{\rho}(Z_0) \to \mathbb{C}$ be defined by (1.27) if $a = \infty$ and by (1.29) if $a \neq \infty$, respectively. Then d_a is analytic, $d_a(a) = 1$, and $\lambda \in \sigma_d(Z)$ if and only if $d_a(\lambda) = 0$.*

Remark 1.5.6. The definitions of d_a and d_∞ are consistent in the following sense: If $Z, Z_0 \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \rho(Z_0)$, then $\lim_{|a|\to\infty} d_a(\lambda) = d_\infty(\lambda)$.[14] This can be seen as follows: If $a \in \rho(Z) \cap \rho(Z_0)$ then (1.28), (1.29) and the second resolvent identity imply that

$$d_a(\lambda) = \det_{\lceil p \rceil}(I - (a-\lambda)R_Z(a)(Z - Z_0)R_{Z_0}(\lambda)).$$

The statement follows from the continuity of the determinant and the fact that $\|(a-\lambda)R_Z(a) - I\| = \|(\lambda - Z)R_Z(a)\| \to 0$ for $|a| \to \infty$. $\square$

We will refer to the function $d_a = d_a^{Z,Z_0}$ as the pth **perturbation determinant** of Z by Z_0 (with index a). Of course, the p-dependence of d_a is neglected in our notation.

Remark 1.5.7. The above definition of perturbation determinants is an extension of the standard one (which coincides with the function d_∞), see, e.g., (Gohberg & Krein 1969) and (Yafaev 1992). $\square$

It still remains to show that the multiplicity of λ_0 as an eigenvalue of Z coincides with its order as a zero of d_a.

[13] The function $\lambda \mapsto (a-\lambda)^{-1}$ maps $\hat{\rho}(Z_0)$ conformally onto $\hat{\rho}(R_{Z_0}(a))$, see Section 2.1.

[14] The proof shows that this convergence is uniform on compact subsets of $\rho(Z_0)$.

Proposition 1.5.8. *Let $a \in \hat{\rho}(Z_0) \cap \hat{\rho}(Z)$ where Z, Z_0 satisfy Assumption 1.5.1, and let $d_a : \hat{\rho}(Z_0) \to \mathbb{C}$ be defined as above. Then the following statements are equivalent:*

(i) $\lambda_0 \in \rho(Z_0)$ *is a zero of* d_a *of order* k_0.

(ii) $\lambda_0 \in \sigma_d(Z)$ *with algebraic multiplicity* $m_Z(\lambda_0) = k_0$.

Remark 1.5.9. Although seemingly well-known, we were not able to find a proof of the above equivalence in the literature (apart from the case $p = 1$ and $a = \infty$, which can be found in (Gohberg & Krein 1969), p.173-174). For this reason, we will provide a full proof. □

The next lemma is one of the main ingredients in the proof of Proposition 1.5.8.

Lemma 1.5.10. *Let $K \in \mathcal{S}_n(\mathcal{H})$, where $n \in \mathbb{N}$, and let $F \in \mathcal{F}(\mathcal{H})$. Then*

$$\det_n((I-F)(I-K)) = e^{\operatorname{tr}(p_n(F,K))} \det(I-F)\det_n(I-K) \tag{1.30}$$

where $p_n(F, K) \in \mathcal{F}(\mathcal{H})$ is a polynomial in F and K.

Remark 1.5.11. We note that the left-hand side of (1.30) is well defined since $(I - F)(I - K) = I - (F + K - FK)$ and $(F + K - FK) \in \mathcal{S}_n(\mathcal{H})$. □

Proof of Lemma 1.5.10. For $m \in \mathbb{N}$ let $P_m \in \mathcal{F}(\mathcal{H})$ denote a sequence of orthogonal[15] projections converging strongly to the identity operator on $\mathcal{H}$. Setting $K_m = P_m K P_m \in \mathcal{F}(\mathcal{H})$ we have

$$\|(F + K_m - FK_m) - (F + K - FK)\|_{\mathcal{S}_n} \overset{m\to\infty}{\to} 0,$$

see (Gohberg et al. 2000), p.89. Using the continuity of the determinant we thus obtain

$$\det_n((I-F)(I-K)) = \lim_{m\to\infty} \det_n(I - (F + K_m - FK_m)). \tag{1.31}$$

Let us note that for $G, H \in \mathcal{F}(\mathcal{H})$, we have

$$\det_n(I-G) = \det(I-G)\exp\left(\sum_{j=1}^{n-1} \frac{\operatorname{tr}(G^j)}{j}\right), \tag{1.32}$$

as follows from definition (1.23) (here we set $\sum_{j=1}^{0}(\dots) := 0$), and

$$\det((I-G)(I-H)) = \det(I-G)\det(I-H),$$

which is certainly true in the finite-dimensional case. Since $F, K_m \in \mathcal{F}(\mathcal{H})$ we thus obtain

$$\begin{aligned}
&\det_n(I - (F + K_m - FK_m)) \\
&= \det(I - (F + K_m - FK_m))\exp\left(\sum_{j=1}^{n-1} \frac{\operatorname{tr}((F + K_m - FK_m)^j)}{j}\right) \\
&= \det(I-F)\det(I-K_m)\exp\left(\sum_{j=1}^{n-1} \frac{\operatorname{tr}((F + K_m - FK_m)^j)}{j}\right).
\end{aligned}$$

[15] A projection $P \in \mathcal{B}(\mathcal{H})$ is called orthogonal if $P = P^*$.

Using (1.32) and the linearity of the trace, the last identity can be rewritten as

$$\begin{aligned}&\det{}_n(I-(F+K_m-FK_m))\\ =\;&\det(I-F)\det{}_n(I-K_m)\exp\left(\sum_{j=1}^{n-1}\frac{\operatorname{tr}((F+K_m-FK_m)^j-K_m^j)}{j}\right).\end{aligned} \tag{1.33}$$

To finish the proof, we first note that, as above, the continuity of the determinant implies that

$$\lim_{m\to\infty}\det{}_n\,(I-K_m)=\det{}_n(I-K). \tag{1.34}$$

Moreover, for $1\leq j\leq n-1$ and $m\to\infty$ we have

$$\begin{aligned}(F+K_m-FK_m)^j-K_m^j &= \sum_{l=0}^{j-1}(F+K_m-FK_m)^{j-1-l}F(I-K_m)K_m^l\\ &\overset{\mathcal{S}_1}{\to} \sum_{l=0}^{j-1}(F+K-FK)^{j-1-l}F(I-K)K^l\in\mathcal{F}(\mathcal{H}).\end{aligned} \tag{1.35}$$

Using the continuity of the trace, and combining (1.35), (1.34), (1.33) and (1.31) we obtain (1.30). ■

The following proof of Proposition 1.5.8 closely follows the proof of the $p=1$ case provided in (Gohberg & Krein 1969).

Proof of Proposition 1.5.8. First, let us note that it is sufficient to prove the result in the case $Z,Z_0\in\mathcal{B}(\mathcal{H})$ and $a=\infty$. The general case is then a consequence of the definition of d_a and Proposition 1.1.6.

Assuming that $Z,Z_0\in\mathcal{B}(\mathcal{H})$ with $Z-Z_0\in\mathcal{S}_p(\mathcal{H})$, we have to show that the order of λ_0 as a zero of d_∞ and its multiplicity as an eigenvalue of Z coincide. To simplify notation we set $n=\lceil p\rceil$.

Let $\lambda_0\in\sigma_d(Z)\subset\rho(Z_0)$ and let $P=P_Z(\lambda_0)$ be the corresponding Riesz projection. It is no loss to assume that $\lambda_0\neq 0$ since we can always replace the pair (Z,Z_0) by the pair $(Z+\varepsilon,Z_0+\varepsilon)$ for some suitable $\varepsilon>0$ and use that

$$P_Z(0)=P_{Z+\varepsilon}(\varepsilon)\quad\text{and}\quad d_\infty^{Z+\varepsilon,Z_0+\varepsilon}(\lambda)=d_\infty^{Z,Z_0}(\lambda-\varepsilon).$$

Let us introduce the operators $R=ZP\in\mathcal{F}(\mathcal{H})$ and $R^\perp=Z(I-P)\in\mathcal{B}(\mathcal{H})$. We note that $\sigma(R)=\{\lambda_0\}$ and that $\lambda_0\in\rho(R^\perp)$. In particular, there exists a ball $U(\lambda_0)$ around λ_0, with $0\notin U(\lambda_0)$, such that $\lambda-Z_0$ and $\lambda-R^\perp$ are invertible for every $\lambda\in U(\lambda_0)$.

In the following, let $\lambda\in U(\lambda_0)$. Since $RR^\perp=R^\perp R=0$ and $Z=R+R^\perp$, we have

$$\begin{aligned}\lambda(\lambda-Z)(\lambda-Z_0)^{-1} &= (\lambda-R)(\lambda-R^\perp)(\lambda-Z_0)^{-1}\\ &= (\lambda-R)[I-(R^\perp-Z_0)(\lambda-Z_0)^{-1}].\end{aligned}$$

Moreover, since $R\in\mathcal{F}(\mathcal{H})$ and

$$(R^\perp-Z_0)(\lambda-Z_0)^{-1}=(Z-Z_0)(\lambda-Z_0)^{-1}-R(\lambda-Z_0)^{-1}\in\mathcal{S}_n(\mathcal{H}),$$

we obtain from Lemma 1.5.10 that

$$\begin{aligned} d_\infty(\lambda) &= \det_n(I-(Z-Z_0)(\lambda-Z_0)^{-1}) = \det_n((\lambda-Z)(\lambda-Z_0)^{-1}) \\ &= \det_n([I-\lambda^{-1}R][I-(R^\perp-Z_0)(\lambda-Z_0)^{-1}]) \\ &= e^{\mathrm{tr}(p_n(\lambda^{-1}R,(R^\perp-Z_0)(\lambda-Z_0)^{-1})}\det(I-\lambda^{-1}R)\det_n(I-(R^\perp-Z_0)(\lambda-Z_0)^{-1}). \end{aligned}$$

We note that the first factor on the right-hand side is analytic on $U(\lambda_0)$ (as a function of λ) and non-vanishing. Moreover, the operator

$$I-(R^\perp-Z_0)(\lambda-Z_0)^{-1} = (\lambda-R^\perp)(\lambda-Z_0)^{-1}$$

is invertible, so the order of λ_0 as a zero of $d_\infty(\lambda)$ coincides with its order as a zero of $\lambda \mapsto \det(I-\lambda^{-1}R)$. Since $R \in \mathcal{F}(\mathcal{H})$ and

$$\det(I-\lambda^{-1}R) = \det(I-\lambda^{-1}ZP) = (1-\lambda^{-1}\lambda_0)^{m_Z(\lambda_0)},$$

where $m_Z(\lambda_0) = \mathrm{Rank}(P)$ is the algebraic multiplicity of λ_0, this concludes the proof of Proposition 1.5.8.
■

We conclude this section with two estimates on the perturbation determinant. The first is a direct consequence of Proposition 1.4.3 and the definition of d_a.

Proposition 1.5.12. *Let $a \in \rho(Z_0) \cap \rho(Z)$ where Z, Z_0 satisfy Assumption 1.5.1, and let $d_a : \hat{\rho}(Z_0) \to \mathbb{C}$ be defined as above. Then for $\lambda \in \hat{\rho}(Z_0)$ we have*

$$|d_a(\lambda)| \leq \exp\left(\Gamma_p \|[R_Z(a)-R_{Z_0}(a)][(a-\lambda)^{-1}-R_{Z_0}(a)]^{-1}\|_{\mathcal{S}_p}^p\right), \qquad (1.36)$$

where Γ_p is the constant which first occurred in Proposition 1.4.3.

For later purposes, we not only formulate an estimate on d_∞ in terms of $\|(Z-Z_0)R_{Z_0}(\lambda)\|_{\mathcal{S}_p}^p$, but we also provide a little more general estimate.

Proposition 1.5.13. *Let $Z, Z_0 \in \mathcal{B}(\mathcal{H})$ satisfy Assumption 1.5.1. Then for $\lambda \in \hat{\rho}(Z_0)$ we have*

$$|d_\infty(\lambda)| \leq \exp\left(\Gamma_p \|(Z-Z_0)R_{Z_0}(\lambda)\|_{\mathcal{S}_p}^p\right). \qquad (1.37)$$

If, in addition, $Z - Z_0 = M_1M_2$ where M_1, M_2 are bounded operators on $\mathcal{H}$ such that $M_2R_{Z_0}(a)M_1 \in \mathcal{S}_p(\mathcal{H})$ for every $a \in \rho(Z_0)$, then for $\lambda \in \hat{\rho}(Z_0)$ we have

$$|d_\infty(\lambda)| \leq \exp\left(\Gamma_p \|M_2R_{Z_0}(\lambda)M_1\|_{\mathcal{S}_p}^p\right). \qquad (1.38)$$

In both cases, Γ_p is the constant which first occurred in Proposition 1.4.3.

Remark 1.5.14. While the non-zero eigenvalues of $M_1M_2R_{Z_0}(a)$ and $M_2R_{Z_0}(a)M_1$ coincide, the same need not be true for their singular values. In particular, while the operator $M_1M_2R_{Z_0}(a) = (Z-Z_0)R_{Z_0}(a)$ is automatically in $\mathcal{S}_p(\mathcal{H})$ when $Z, Z_0 \in \mathcal{B}(\mathcal{H})$ satisfy Assumption 1.5.1, in general this need not imply that $M_2R_{Z_0}(a)M_1 \in \mathcal{S}_p(\mathcal{H})$ as well. □

Proof of Proposition 1.5.13. Estimate (1.38) is a consequence of Proposition 1.4.3, the definition of d_∞ and the identity

$$\det_{\lceil p \rceil}(I - (Z - Z_0)R_{Z_0}(\lambda)) = \det_{\lceil p \rceil}(I - M_1 M_2 R_{Z_0}(\lambda)) = \det_{\lceil p \rceil}(I - M_2 R_{Z_0}(\lambda) M_1),$$

which follows from Proposition 1.4.1 since $M_2 R_{Z_0}(\lambda) M_1 \in \mathcal{S}_p(\mathcal{H})$ by assumption. Estimate (1.37) follows from (1.38) choosing $M_1 = I$ and $M_2 = Z - Z_0$. ∎

2. Zeros of holomorphic functions on the unit disk

We have seen in Section 1.5 that the discrete spectrum of a linear operator Z satisfying Assumption 1.5.1 coincides with the zero set of the corresponding perturbation determinant. Moreover, this determinant is a holomorphic function which grows exponentially near the boundary of its domain of definition. In this chapter, to obtain further information on the discrete spectrum of Z, we will discuss the general problem of relating the growth properties of a holomorphic function (on the unit disk) to the distribution of its zeros.

2.1. A short motivation

Summary: Motivated by a discussion of typical growth estimates on the perturbation determinant, we introduce a class of holomorphic functions on the unit disk satisfying a non-radial exponential growth bound. Moreover, we provide a first estimate on the distribution of zeros of these functions.

Let us consider the following situation: $Z_0 \in \mathcal{B}(\mathcal{H})$ is a selfadjoint operator with

$$\sigma(Z_0) = \sigma_{ess}(Z_0) = [\zeta_1, \zeta_2] \,\dot{\cup}\, [\zeta_3, \zeta_4] \,\dot{\cup}\, \ldots \,\dot{\cup}\, [\zeta_{N-1}, \zeta_N], \tag{2.1}$$

where $\zeta_1 < \zeta_2 < \ldots < \zeta_N$, and $Z = Z_0 + M$ where $M \in \mathcal{S}_p(\mathcal{H})$ for some fixed $p > 0$. By Remark 1.2.9, given these assumptions the spectrum of Z can differ from the spectrum of Z_0 by an at most countable sequence of eigenvalues of finite type (constituting the discrete spectrum of Z), which can accumulate on the spectrum of Z_0 only.

Since $\sigma_d(Z)$ is precisely[1] the zero set of the pth perturbation determinant $d = d_\infty^{Z,Z_0}$, it should be possible to obtain further information on the distribution of the eigenvalues of Z by studying the analytic function d, in particular, by taking advantage of the estimate provided on d in Proposition 1.5.13, i.e.,

$$|d(\lambda)| \leq \exp\left(\Gamma_p \|MR_{Z_0}(\lambda)\|_{\mathcal{S}_p}^p\right), \qquad \lambda \in \hat{\rho}(Z_0). \tag{2.2}$$

Taking a closer look at this estimate we note that, in general, the right-hand side of (2.2) will explode when λ approaches the spectrum of Z_0. For instance, using estimate (1.18) and the identity

$$\|R_{Z_0}(\lambda)\| = [\text{dist}(\lambda, \sigma(Z_0))]^{-1}, \tag{2.3}$$

[1] Z and Z_0 satisfy Assumption 1.5.1 by Remark 1.2.9.

which is valid since Z_0 is selfadjoint, we obtain

$$|d(\lambda)| \leq \exp\left(\frac{\Gamma_p \|M\|_{\mathcal{S}_p}^p}{\operatorname{dist}(\lambda, \sigma(Z_0))^p}\right). \tag{2.4}$$

In this case, the rate of explosion of the right-hand side of (2.4) is the same for λ approaching an interior point of $\sigma(Z_0)$ and for λ approaching one of the boundary points ζ_j. However, as we will see in subsequent chapters, for more concrete operators the situation can be different and a more precise analysis of the $\mathcal{S}_p$-norm of $MR_{Z_0}(\lambda)$ will lead to an estimate of the form

$$\|MR_{Z_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{C(p,M)}{\operatorname{dist}(\lambda, \sigma(Z_0))^{\alpha'} \prod_k \operatorname{dist}(\lambda, \zeta_k)^{\beta_k'}}, \tag{2.5}$$

where α' and β_k' are some non-negative parameters with $\alpha' + \sum_k \beta_k' = p$. Of course, estimate (2.4) would change accordingly.

To get an idea why such a precise knowledge of the growth behavior of d might help us to obtain additional information on the distribution of its zeros, and hence on the distribution of the eigenvalues of Z, let us continue with some general remarks concerning holomorphic functions on the unit disk.

We start with some notations: The class of all complex-valued functions defined and holomorphic on an open set $\Omega \subset \hat{\mathbb{C}}$ is denoted by $H(\Omega)$. If $f \in H(\Omega)$ then

$$\mathcal{Z}(f) = \{\lambda \in \Omega : f(\lambda) = 0\} \tag{2.6}$$

denotes the corresponding **zero set**. By $H^\infty(\Omega)$ we denote the class of all bounded holomorphic functions on Ω. Moreover, $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ denote the **unit disk** and **unit circle**, respectively.

It is well known that the zero set of a non-trivial function $h \in H(\mathbb{D})$ is discrete, with possible accumulation points on the boundary $\mathbb{T}$. In other words, $\mathcal{Z}(h)$ is either finite, or it can be written as $\mathcal{Z}(h) = \{w_k\}_{k=1}^\infty$, where $|w_k|$ is strictly increasing and

$$\lim_{k\to\infty} (1 - |w_k|) = 0. \tag{2.7}$$

While in this generality nothing more can be said about $\mathcal{Z}(h)$, the situation changes drastically if we assume that $h \in H^\infty(\mathbb{D})$. In this case, the sequence of zeros of h does not only satisfy (2.7) but the much stronger **Blaschke condition**

$$\sum_{w \in \mathcal{Z}(h)} (1 - |w|) < \infty, [2] \tag{2.8}$$

where each zero of h is counted according to its multiplicity (meaning that a zero of order m is counted exactly m-times). This estimate shows that the convergence of the zeros of h to the boundary of $\mathbb{D}$ must occur with a sufficiently high "speed". Comparing

[2] A proof of (2.8) will be given in Proposition 2.2.8 below.

(2.7) and (2.8) we receive a first impression of the large influence of the growth behavior of a holomorphic function on the distribution of its zeros.

Guided from the previous considerations, let us continue with our discussion of the perturbation determinant d, which, as we recall, was defined on $\hat{\rho}(Z_0)$. To this end, let us assume that $\Omega \subset \hat{\rho}(Z_0)$ is **conformally equivalent** to the unit disk, i.e., there exists a bijective continuous mapping $\phi : \mathbb{D} \to \Omega$ which is holomorphic on $\{w \in \mathbb{D} : \phi(w) \neq \infty\}$ (in this case ϕ is called a **conformal mapping**[3] of $\mathbb{D}$ onto Ω). In addition, let us assume that $\partial\Omega \cap \sigma(Z_0) \neq \emptyset$. In particular, this last assumption implies that $\Omega \cap \sigma_d(Z)$ is either finite, or it consists of a sequence of eigenvalues of Z which can accumulate on $\partial\Omega \cap \sigma(Z_0)$ only.

Remark 2.1.1. If $\Omega \subset \mathbb{C}$ then we consider $\partial\Omega$ as a subset of $\mathbb{C}$ as well. In general, since Ω is conformally equivalent to the unit disk, the point ∞ can only be contained in Ω as an interior point, so in any case $\partial\Omega$ is a subset of the plane. □

For simplicity, let us suppose that $\infty \in \Omega$ and that $\phi(0) = \infty$. The conformality of ϕ implies that $|\phi^{-1}(\lambda_k)| \to 1$ for every sequence $\{\lambda_k\} \subset \Omega$ that converges to a point on $\partial\Omega$. Moreover, $\lambda \in \Omega$ is a zero of d if and only if $w = \phi^{-1}(\lambda)$ is a zero of $d \circ \phi$ of the same order. Thus, to obtain information on the zero set of d in Ω, and hence on the discrete spectrum of Z in this set, it is sufficient to study the zero set of the function $h_0 := d \circ \phi$, which is analytic on $\mathbb{D}$ (while h_0 is obviously analytic on $\mathbb{D} \setminus \{0\}$, we can extend it to an analytic function on $\mathbb{D}$ noting that $\lim_{\lambda \to 0} h_0(\lambda) = d(\infty) = 1$).

As indicated by the estimates (2.2) and (2.5) above, the functions h_0 arising in this manner will obey a special growth estimate. Namely, $h_0(0) = 1$ and

$$|h_0(w)| \leq \exp\left(\frac{K|w|^\gamma}{(1-|w|)^\alpha \prod_{j=1}^N |w - \xi_j|^{\beta_j}} \right), \quad w \in \mathbb{D}. \tag{2.9}$$

Here $\xi_j \in \mathbb{T}$ (corresponding to the boundary points $\zeta_j \in \sigma(Z_0)$), and α, β_j, γ and K are non-negative.[4] In other words, these functions explode exponentially on the boundary of $\mathbb{D}$, with the explosion being due to some radial part and finitely many additional singularities. Since this class of functions will be of importance in the sequel, an explicit definition is in order. To this end, let us set $\mathbb{R}_+ = [0, \infty)$ and

$$(\mathbb{T}^N)_* = \{(\xi_1, \dots, \xi_N) \in \mathbb{T}^N : \xi_i \neq \xi_j, 1 \leq i < j \leq N\}, \quad N \in \mathbb{N}. \tag{2.10}$$

Definition 2.1.2. Let $\alpha, \gamma, K \in \mathbb{R}_+$. For $N \in \mathbb{N}$ let $\vec{\beta} = (\beta_1, \dots, \beta_N) \in \mathbb{R}_+^N$ and $\vec{\xi} = (\xi_1, \dots, \xi_N) \in (\mathbb{T}^N)_*$. The class of all functions $h \in H(\mathbb{D})$ satisfying $h(0) = 1$ and obeying (2.9) (for this choice of parameters) is denoted by $\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$. Moreover, we set $\mathcal{M}(\alpha, K) = \mathcal{M}(\alpha, \vec{0}, 0, \vec{\xi}, K)$.[5] □

[3] This definition of conformal mappings coincides with the standard one if $\Omega \subset \mathbb{C}$. A set $\Omega \subset \hat{\mathbb{C}}$ is conformally equivalent to the unit disk if it is open and the complement of Ω with respect to the Riemann sphere is connected and consists of more than one point. For a more detailed discussion of conformal mappings in the extended plane we refer to (Palka 1991), Chapter IX.1.6.

[4] We note that the parameters α and β_j will generally be different from the corresponding parameters α' and β_j' occurring in estimate (2.5).

[5] Clearly, the right-hand side in this definition is independent of $\vec{\xi}$.

Convention 2.1.3. Throughout this chapter, whenever speaking of $\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$ we will always implicitly assume that the parameters are chosen as indicated in the previous definition. □

Remark 2.1.4. We have the inclusions

$$\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K) \subset \mathcal{M}(\alpha', \vec{\beta}, \gamma', \vec{\xi}, K')$$

if $\alpha \leq \alpha', \gamma \geq \gamma'$ and $K \leq K'$, and

$$\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K) \subset \mathcal{M}(\alpha, \vec{\beta}', \gamma, \vec{\xi}, K \cdot 2^{\sum_j \beta'_j})$$

if $\beta_j \leq \beta'_j$. □

To give a first illustration how the bound (2.9) can be used to obtain additional information on the discrete spectrum of Z, let us consider the one case that we can handle so far, that is, let us assume that h_0 is bounded (meaning that the exponents α and β_j in (2.9) are all equal to zero). In this case, the Blaschke condition (2.8) and our previous discussion imply that

$$\sum_{\lambda \in \sigma_d(Z) \cap \Omega} (1 - |\phi^{-1}(\lambda)|) = \sum_{w \in \mathcal{Z}(h_0)} (1 - |w|) < \infty. \tag{2.11}$$

This estimate increases our knowledge about the discrete spectrum of Z considerably, showing that if there are infinitely many eigenvalues of Z in Ω, then these eigenvalues have to converge to the spectrum of Z_0 sufficiently fast. More precisely, in the next chapter we will see that the summands on the left-hand side of (2.11), for λ in the vicinity of $\sigma(Z_0)$, behave like $\operatorname{dist}(\lambda, \sigma(Z_0))|\phi'(\phi^{-1}(\lambda))|^{-1}$ so (2.11) implies that

$$\sum_{\lambda \in \sigma_d(Z) \cap \Omega} \frac{\operatorname{dist}(\lambda, \sigma(Z_0))}{|\phi'(\phi^{-1}(\lambda))|} < \infty. \tag{2.12}$$

In this way, we obtain a detailed estimate on the distribution of the eigenvalues of Z.

Example 2.1.5. Let us demonstrate the content of information of (2.12) for the special case when $\sigma(Z_0) = [\zeta_1, \zeta_2]$ and $\Omega = \hat{\mathbb{C}} \setminus [\zeta_1, \zeta_2]$: In this case, as we will see in the next chapter, (2.12) takes the form

$$\sum_{\lambda \in \sigma_d(Z)} \frac{\operatorname{dist}(\lambda, [\zeta_1, \zeta_2])}{|\lambda - \zeta_1|^{1/2} |\lambda - \zeta_2|^{1/2}} < \infty. \tag{2.13}$$

The finiteness of this sum has consequences regarding sequences $\{\lambda_k\}$ of isolated eigenvalues of Z converging to some $\lambda^* \in [\zeta_1, \zeta_2]$. Taking a subsequence, we can suppose that one of the following options holds:

(i.a) $\lambda^* = \zeta_1$ and $\operatorname{Re}(\lambda_k) \leq \zeta_1$. (i.b) $\lambda^* = \zeta_2$ and $\operatorname{Re}(\lambda_k) \geq \zeta_2$.

(ii.a) $\lambda^* = \zeta_1$ and $\operatorname{Re}(\lambda_k) > \zeta_1$. (ii.b) $\lambda^* = \zeta_2$ and $\operatorname{Re}(\lambda_k) < \zeta_2$.

(iii) $\lambda^* \in (\zeta_1, \zeta_2)$.

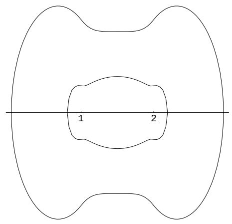

Figure 2.1.: A sketch of $\partial\Omega_\varepsilon$ with $\varepsilon = 0.7$ and $\varepsilon = 0.4$, respectively.

It is sufficient to consider the cases (i.a), (ii.a) and (iii) only. Since

$$\operatorname{dist}(\lambda, [\zeta_1, \zeta_2]) = \begin{cases} |\lambda - \zeta_1|, & \text{if } \operatorname{Re}(\lambda) < \zeta_1, \\ |\operatorname{Im}(\lambda)|, & \text{if } \zeta_1 \leq \operatorname{Re}(\lambda) \leq \zeta_2, \\ |\lambda - \zeta_2|, & \text{if } \operatorname{Re}(\lambda) > \zeta_2, \end{cases}$$

in case (i.a) estimate (2.13) implies the finiteness of $\sum_k |\lambda_k - \zeta_1|^{1/2}$ showing that any such sequence must converge to ζ_1 sufficiently fast. Similarly, in case (ii.a) estimate (2.13) implies the finiteness of $\sum_k \frac{|\operatorname{Im}(\lambda_k)|}{|\lambda_k - \zeta_1|^{1/2}}$. Finally, in case (iii) we obtain the finiteness of $\sum_k |\operatorname{Im}(\lambda_k)|$, showing that the sequence must converge to the real line sufficiently fast.

Estimate (2.13) has consequences on the number of eigenvalues of Z as well. For instance, if

$$\Omega_\varepsilon = \left\{ \lambda \in \mathbb{C} \setminus [\zeta_1, \zeta_2] : \frac{\operatorname{dist}(\lambda, [\zeta_1, \zeta_2])}{|\lambda - \zeta_1|^{1/2} |\lambda - \zeta_2|^{1/2}} > \varepsilon \right\},$$

see Figure 2.1, and $N(Z, \Omega_\varepsilon)$ denotes the number of eigenvalues of Z in this set, then (2.13) implies that $N(Z, \Omega_\varepsilon) = O(\varepsilon^{-1})$ for $\varepsilon \downarrow 0$, i.e., $N(Z, \Omega_\varepsilon)$ does not grow faster than ε^{-1}. $\square$

Unfortunately, the functions h_0 arising in applications will generally not be bounded, so the Blaschke condition (2.8) cannot be used in the study of their zero sets. For this reason, in the subsequent sections of the present chapter we will derive estimates on the zero sets of functions $h \in \mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$, which modify and extend condition (2.8). These estimates will then be used in the following chapters to derive estimates on the discrete spectrum of bounded and unbounded linear operators, in a similar fashion as described above.

Our main estimate on the distribution of zeros of functions in the class $\mathcal{M}$ will be presented in Section 2.4. It is based on a recent result by Borichev et al. (2009). From our point of view, the most accessible proof of their result, and the one that we will sketch below, is based on results from potential theory in the complex plane, transferring the problem to one about subharmonic functions, see (Favorov & Golinskii 2009). Although not necessary on logical grounds, we think that, instead of presenting this result immediately, it is worthwhile to start with a presentation of some more elementary results, which will then turn out to be special cases of the theorem presented in Section 2.4, and which require only a modest background in complex function theory.

Since the problem of relating the growth properties of a holomorphic function to the distribution of its zeros is a well-studied topic in complex analysis, we will start with a look at some of the classical results in this field and analyze their applicability to our problem. This will be done in Section 2.2. In Section 2.3 we will present some first extensions of these classical results, based on joint work with M. Demuth and G. Katriel. Having completed these two "introductory" sections, we will be prepared to study the most general results in Section 2.4.

Remark 2.1.6. The results of Section 2.2 and 2.3 will not be used in the chapters to follow so the impatient reader may skip these sections and go directly to Section 2.4. □

2.2. Classical results

Summary: We present some classical results on the distribution of zeros of functions in the class $\mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$.[6] The consideration of the class $\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$, with $\gamma \neq 0$, is deferred to the following section.

Convention 2.2.1. Let us agree that, throughout this thesis, whenever a sum involving zeros of a holomorphic function is considered, each zero is counted according to its multiplicity. □

We begin our discussion with **Jensen's identity** which serves as a main tool in connecting the growth properties of a holomorphic function with the distribution of its zeros. To this end, let us set $\mathbb{D}_r = \{w \in \mathbb{C} : |w| < r\}$ and let us denote the number of zeros of a function h in $\overline{\mathbb{D}}_r$ (counted according to their multiplicities) by $N(h, r)$.

Lemma 2.2.2. *Let $h \in H(\mathbb{D})$ with $|h(0)| = 1$. Then for $r \in (0, 1)$ we have*

$$\int_0^r \frac{N(h,s)}{s} \, ds = \sum_{w \in \mathcal{Z}(h), |w| \leq r} \log \left| \frac{r}{w} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| \, d\theta. \tag{2.14}$$

While the first equality in (2.14) is elementary, it is the second one that is usually referred to as Jensen's identity.

Remark 2.2.3. For the convenience of the reader we will provide a proof (or at least a sketch of proof) for every result to be discussed in this chapter. □

*Sketch of proof of Lemma 2.2.2 (see (*Rudin *1987), p.308).* Let $w_1, \ldots, w_m$ denote the zeros of h in $\mathbb{D}_r$ (multiplicity taken into account) and let $w_{m+1}, \ldots, w_N$ denote the zeros of h of modulus r. For $\varepsilon > 0$ sufficiently small the function

$$g(w) = h(w) \prod_{n=1}^{m} \frac{r^2 - \overline{w}_n w}{r(w_n - w)} \prod_{n=m+1}^{N} \frac{w_n}{w_n - w} \tag{2.15}$$

[6] Readers interested in the related (and much more extensive) value distribution theory of entire or meromorphic functions might want to take a look at (Nevanlinna 1953). For results on holomorphic functions on the unit disk which grow polynomially, rather than exponentially, we refer to (Duren & Schuster 2004).

is holomorphic and non-zero in the disk $\mathbb{D}_{r+\varepsilon}$. In particular, since $\log|g|$ is harmonic in this disk, the mean value property[7] implies that

$$\sum_{n=1}^{m} \log\left|\frac{r}{w_n}\right| = \log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|d\theta. \tag{2.16}$$

For $1 \le n \le m$ the factors in (2.15) are of absolute value 1 if $|w| = r$. Hence, setting $w_n = re^{i\theta_n}$ for $m+1 \le n \le N$, definition (2.15) shows that

$$\log|g(re^{i\theta})| = \log|h(re^{i\theta})| - \sum_{n=m+1}^{N} \log|1 - e^{i(\theta-\theta_n)}|.$$

Since the integral of $\theta \mapsto \log|1 - e^{i\theta}|$ over the interval $[0, 2\pi]$ vanishes, we can replace g by h on the right-hand side of (2.16) without changing the integral. This concludes the proof. ∎

Remark 2.2.4. If $h \in H(\mathbb{D})$ and $|h(0)| = 1$, then Jensen's identity shows that

$$f : r \mapsto \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\, d\theta$$

is non-negative and non-decreasing on $(0, 1)$. Moreover, if h is non-zero in the disk $\mathbb{D}_{r_0}$, then $f(r) = 0$ for $r \le r_0$. □

We continue with a first extension of the Blaschke condition (2.8). As we will see, (2.8) is a necessary condition on $\mathcal{Z}(h)$ not only for $h \in H^\infty(\mathbb{D})$ but for a much broader class of functions.

Proposition 2.2.5. *Let $h \in H(\mathbb{D})$ with $|h(0)| = 1$. Suppose that*

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta = K_0 < \infty. \tag{2.17}$$

Then $\mathcal{Z}(h)$ satisfies the Blaschke condition

$$\sum_{w\in\mathcal{Z}(h)} (1 - |w|) \le K_0. \tag{2.18}$$

Proof. The statement follows from Jensen's identity, Remark 2.2.4 and the observation that

$$\sum_{w\in\mathcal{Z}(h)} (1 - |w|) \le \sum_{w\in\mathcal{Z}(h)} (-\log|w|) = \int_0^1 \frac{N(h,s)}{s}\, ds.$$

∎

[7] A short review of harmonic functions is provided in Appendix A.

Remark 2.2.6. The following partial converse of the last proposition is true: Given a discrete set $\mathcal{Z} \subset \mathbb{D}$ satisfying condition (2.18) (with $\mathcal{Z}(h)$ replaced by $\mathcal{Z}$) there exists a function $g \in H^\infty(\mathbb{D})$ with $\mathcal{Z}(g) = \mathcal{Z}$, see (Rudin 1987), p.310. In particular, choosing $\mathcal{Z} = \{w_n\}_{n\in\mathbb{N}}$ such that the sequence $r_n = 1 - |w_n|$ is in $l^1(\mathbb{N})$ but not in $l^p(\mathbb{N})$ for any $p < 1$, we see that for bounded holomorphic functions (2.18) is best possible, i.e., there is no $p < 1$ such that $\sum_{w\in\mathcal{Z}(h)}(1-|w|)^p < \infty$ for every $h \in H^\infty(\mathbb{D})$. □

The next lemma will help us classify those $h \in \mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$ which satisfy condition (2.17).

Lemma 2.2.7. *Let $\xi \in \mathbb{T}$ and $r \in (0,1)$. Then for $\beta \geq 0$ and $r \to 1$ we have*

$$\int_0^{2\pi} \frac{d\theta}{|re^{i\theta} - \xi|^\beta} = \begin{cases} O\left(\frac{1}{(1-r)^{\beta-1}}\right), & \text{if } \beta > 1, \\ O\left(-\log(1-r)\right), & \text{if } \beta = 1, \\ O(1), & \text{if } \beta < 1. \end{cases} \tag{2.19}$$

Proof. It is sufficient to consider the case $\xi = 1$. Furthermore, we only have to show that for $\frac{1}{2} < r < 1$ and $r \to 1$ the integral

$$\int_0^{\pi/2} \frac{d\theta}{|re^{i\theta} - 1|^\beta} = \int_0^{\pi/2} \frac{d\theta}{(r^2 - 2r\cos(\theta) + 1)^{\frac{\beta}{2}}}$$

behaves in the way indicated in (2.19).

For $0 < \theta < \pi/2$ we have $\cos(\theta) \leq 1 - \frac{\theta^2}{4}$ so

$$(r^2 - 2r\cos(\theta) + 1)^{\beta/2} \geq \left((1-r)^2 + r\frac{\theta^2}{2}\right)^{\beta/2} \geq C_\beta \left(1 - r + \sqrt{r}\theta\right)^\beta .$$

Noting that

$$\frac{1}{(1-r+\sqrt{r}\theta)^\beta} = \begin{cases} \frac{d}{d\theta}\left(\frac{(1-r+\sqrt{r}\theta)^{1-\beta}}{(1-\beta)\sqrt{r}}\right), & \text{if } \beta \neq 1, \\ \frac{d}{d\theta}\left(\frac{\log(1-r+\sqrt{r}\theta)}{\sqrt{r}}\right), & \text{if } \beta = 1, \end{cases}$$

concludes the proof. ■

Proposition 2.2.8. *If $h \in \mathcal{M}(0, \vec{\beta}, 0, \vec{\xi}, K)$, where $\vec{\beta} = (\beta_1, \ldots, \beta_N)$ and $\max_j \beta_j < 1$, then*

$$\sum_{w\in\mathcal{Z}(h)} (1 - |w|) \leq C(\vec{\beta}, \vec{\xi})K. \tag{2.20}$$

Proof. Let $0 < r < 1$. By assumption, we have

$$\log|h(re^{i\theta})| \leq \frac{K}{\prod_{j=1}^N |re^{i\theta} - \xi_j|^{\beta_j}} \leq C(\vec{\beta}, \vec{\xi})K \sum_{j=1}^N \frac{1}{|re^{i\theta} - \xi_j|^{\beta_j}},$$

where the second inequality is easily justified (recall that by definition $\xi_i \neq \xi_j$ if $i \neq j$). An application of Lemma 2.2.7 and Proposition 2.2.5 concludes the proof. ■

Remark 2.2.9. In applications, the parameters $\alpha, \vec{\beta}, \gamma$ and $\vec{\xi}$ will usually depend on the perturbed operator $Z = Z_0 + M$ only implicitly or not at all. On the contrary, the parameter K will depend on Z, or rather M, in a very explicit way. For instance, as we have seen in the previous section, we may have $K = \Gamma_p \|M\|_{\mathcal{S}_p}^p$. We have therefore provided, and will do so in the following, the precise dependence of the right-hand side of (2.20) on K. □

Remark 2.2.10. Although possible in principle, we will not present the exact values of the constants occurring in the estimates in this thesis. However, we will always carefully indicate the parameters that a constant depends on. □

While the last proposition provides an estimate on $\mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$ when there is no radial explosion ($\alpha = 0$), the next result we want to discuss deals with functions satisfying an estimate of the type

$$|h(w)| \leq \exp\left(\frac{K}{(1-|w|)^\alpha}\right), \quad w \in \mathbb{D}.^8 \tag{2.21}$$

Theorem 2.2.11. *Let $h \in \mathcal{M}(\alpha, K)$ where $\alpha > 0$. Then for every $\tau > 0$ we have*

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|)^{1+\alpha+\tau} < \infty. \tag{2.22}$$

For a proof we refer to (Beller 1977).

Remark 2.2.12. Beller did not provide a specific bound on the sum in (2.22) although a short inspection of his proof shows that this is possible in principle. We do not present the proof because a more general result will be provided in the next section, see Theorem 2.3.3 below. □

We emphasize that the exponent $\alpha + 1$ in (2.22) is best possible in the following sense: for every $\tau > 0$ there exists a function $h \in \mathcal{M}(\alpha, K)$ such that

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|)^{1+\alpha-\tau} = \infty,$$

see the remark after Corollary 2 in (Beller 1977). On the other hand, the question whether (2.22) remains true for $\tau = 0$ seems to be still open (of course, if $\alpha = 0$ then we know that $\tau = 0$ is allowed by Proposition 2.2.8).

Remark 2.2.13. There exists an interesting extension of Theorem 2.2.11 for functions $h \in \mathcal{M}(\alpha, K)$ with $0 < \alpha < 1$. In this case, those zeros $\{w_k\}$ of h that lie on a single radius $re^{i\theta_0}$ satisfy the Blaschke condition $\sum_k (1-|w_k|) < \infty$. Moreover, this result can be further extended to sequences of zeros approaching a point

[8] The first results on the distribution of zeros of functions in this class are due to Nevanlinna (1953). For a more "recent" discussion we refer the works of Djrbashian and collaborators, see, e.g., (Djrbashian & Shamoian 1988).

$\xi \in \mathbb{T}$ from within a **Stolz angle** $S_{\xi,s} = \{w \in \mathbb{D} : |1 - \overline{\xi}w| \leq s(1 - |w|)\}$, where $s \geq 1$. In other words, these sequences have to converge to ξ in a non-tangential manner. We refer to (Shapiro & Shields 1962), (Hayman & Korenblum 1980) and (Shvedenko 2001) for further information. □

So far, we presented estimates on functions $h \in \mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$ with $\alpha = 0$ (no radial explosion) or $\vec{\beta} = \vec{0}$ (a purely radial explosion), respectively. In the following we would like to present a first result on the general case. To this end, let us note that we can always find a constant $C(\vec{\beta}, \vec{\xi})$ such that

$$C(\vec{\beta}, \vec{\xi})(1 - |w|)^{\alpha} \prod_{j=1}^{N} |w - \xi_j|^{\beta_j} \geq (1 - |w|)^{\alpha + \max_j \beta_j}, \quad w \in \mathbb{D}.^9 \tag{2.23}$$

In particular, this estimate implies the inclusion

$$\mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K) \subset \mathcal{M}(\alpha + \max_j \beta_j, C(\vec{\beta}, \vec{\xi})K), \tag{2.24}$$

which allows us to combine Proposition 2.2.8 and Theorem 2.2.11 to obtain the following estimate on functions in $\mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$.

Corollary 2.2.14. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$. Then for every $\tau > 0$ we have*

$$\sum_{w \in \mathcal{Z}(h)} (1 - |w|)^{1 + \alpha + \max_j \beta_j + \tau} < \infty. \tag{2.25}$$

Furthermore, if $\alpha = 0$ and $\max \beta_j < 1$, then $\sum_{w \in \mathcal{Z}(h)} (1 - |w|) \leq C(\vec{\beta}, \vec{\xi})K$.

It should not come by surprise that the described method of reducing a "mixed" problem to a purely radial one does not lead to the best possible estimates on the zero set of h, since a lot of information on the function gets lost in the process. Indeed, in the next section we will see that the last corollary can be improved considerably by differentiating properly between the behavior of h at generic points and at the singular points ξ_j.

2.3. Further consequences of Jensen's identity

Summary: We use Jensen's identity to improve the estimates derived in the previous section. In particular, we provide a first discussion of the class $\mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$ with $\gamma \neq 0$.

This section is based on results from the joint works (Demuth et al. 2008) and (Hansmann & Katriel 2009).

[9] Here we use again that, by assumption, $\xi_i \neq \xi_j$ for $i \neq j$.

We begin this section by noting that the proof of Proposition 2.2.5 showed the validity of the following identity: for $h \in H(\mathbb{D})$ with $|h(0)| = 1$ we have

$$\sum_{w \in \mathcal{Z}(h)} (-\log|w|) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log|h(re^{i\theta})| d\theta. \tag{2.26}$$

With the next proposition we will provide a useful generalization of this identity. To this end, let us denote the *support* of a function $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$ by $\operatorname{supp}(f)$, i.e.,

$$\operatorname{supp}(f) = \overline{\{x \in (a,b) : f(x) \neq 0\}}. \tag{2.27}$$

Moreover, by $f_+ = \max(f,0)$ and $f_- = -\min(f,0)$ we denote the **positive** and **negative parts** of f, respectively (note that we will use the same notation for the positive and negative parts of a real number as well). In addition, we denote the class of all twice-differentiable functions on (a,b) whose second derivative is continuous by $C^2(a,b)$.

Proposition 2.3.1. *Let $\varphi \in C^2(0,1)$ be non-negative and non-increasing with $\lim_{r\to 1}\varphi(r) = \lim_{r\to 1}\varphi'(r) = 0$, $\operatorname{supp}([r\varphi'(r)]')_- \subset [0,1)$ and $\sup_{0<r<1}([r\varphi'(r)]')_- < \infty$. If $h \in H(\mathbb{D})$, with $|h(0)| = 1$, then*

$$\sum_{w \in \mathcal{Z}(h)} \varphi(|w|) = \frac{1}{2\pi} \int_0^1 dr\, [r\varphi'(r)]' \int_0^{2\pi} d\theta \log|h(re^{i\theta})|.^{10} \tag{2.28}$$

Remark 2.3.2. We are mainly interested in the choice $\varphi(r) = (1-r)^q$, with $q > 1$; other possible choices are $\varphi(r) = (-\log(r))^q$ and $\varphi(r) = (r^{-1} - r)^q$, respectively. □

Proof of Proposition 2.3.1. Let $0 < r < 1$. We restate Jensen's identity:

$$\int_0^r ds \frac{N(h,s)}{s} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \log|h(re^{i\theta})|. \tag{2.29}$$

Multiplying both sides of (2.29) by $[r\varphi'(r)]'$ and integrating with respect to r leads to

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^1 dr\, [r\varphi'(r)]' \int_0^{2\pi} d\theta\, \log|h(re^{i\theta})| \\
=& \int_0^1 dr\, [r\varphi'(r)]' \int_0^r ds \frac{N(h,s)}{s} \overset{(\star)}{=} \int_0^1 ds \frac{N(h,s)}{s} \int_s^1 dr\, [r\varphi'(r)]' \\
=& -\int_0^1 ds\, \varphi'(s) N(h,s) = \int_0^\infty dt \left[\frac{d}{dt}\varphi(e^{-t})\right] N(h,e^{-t}).
\end{aligned} \tag{2.30}$$

The application of Fubini's theorem in $(\star)$ is justified by the assumptions made on φ. We can reformulate the right-hand side of the last equation as follows

$$\begin{aligned}
&\int_0^\infty dt \left[\frac{d}{dt}\varphi(e^{-t})\right] N(h,e^{-t}) = \int_0^\infty dt \sum_{w \in \mathcal{Z}(h), |w| < e^{-t}} \left[\frac{d}{dt}\varphi(e^{-t})\right] \\
=& \sum_{w \in \mathcal{Z}(h)} \int_0^{-\log|w|} dt \left[\frac{d}{dt}\varphi(e^{-t})\right] = \sum_{w \in \mathcal{Z}(h)} \varphi(|w|).
\end{aligned}$$

The last equation together with (2.30) yields the result. ■

[10] Of course, both sides of (2.28) may be simultaneously divergent.

The last proposition is the main ingredient in the proof of the following improvement of Corollary 2.2.14.

Theorem 2.3.3. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$. Then for every $\tau > 0$ we have*

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|)^{1+\alpha+\max_j(\beta_j-1)_+ +\tau} \leq C(\alpha, \vec{\beta}, \vec{\xi}, \tau) K. \tag{2.31}$$

Furthermore, if $\alpha = 0$ and $\max_j \beta_j < 1$, then

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|) \leq C(\vec{\beta}, \vec{\xi}) K. \tag{2.32}$$

Proof. Inequality (2.32) is already included in Corollary 2.2.14, so it remains to prove (2.31). For $q > 1$ let $\varphi(r) = (1-r)^q$. Since

$$[r\varphi'(r)]' = q(1-r)^{q-2}(rq-1)$$

we obtain from Proposition 2.3.1 and our assumptions, using that $\int_0^{2\pi} \log|h(re^{i\theta})| d\theta$ is non-negative,

$$\begin{aligned}
\sum_{w \in \mathcal{Z}(h)} (1-|w|)^q &= \frac{q}{2\pi} \int_0^1 dr \frac{(rq-1)}{(1-r)^{2-q}} \int_0^{2\pi} d\theta \log|h(re^{i\theta})| \\
&\leq \frac{q}{2\pi} \int_{1/q}^1 dr \frac{(rq-1)}{(1-r)^{2-q}} \int_0^{2\pi} d\theta \, \log|h(re^{i\theta})| \\
&\leq \frac{Kq(q-1)}{2\pi} \int_{1/q}^1 dr \frac{1}{(1-r)^{2-q+\alpha}} \int_0^{2\pi} \frac{d\theta}{\prod_{j=1}^N |re^{i\theta}-\xi_j|^{\beta_j}} \\
&\leq \frac{KC(\vec{\beta}, \vec{\xi}) q(q-1)}{2\pi} \sum_{j=1}^N \int_{1/q}^1 dr \frac{1}{(1-r)^{2-q+\alpha}} \int_0^{2\pi} \frac{d\theta}{|re^{i\theta}-\xi_j|^{\beta_j}},
\end{aligned} \tag{2.33}$$

where for the last estimate we used that, by definition, $\xi_i \neq \xi_j$ if $i \neq j$. By Lemma 2.2.7 the right-hand side of (2.33) is finite whenever $q > 1 + \alpha + \max_j(\beta_j - 1)_+$. Choosing $q = 1 + \alpha + \max_j(\beta_j-1)_+ + \tau$ concludes the proof. ∎

To see that the previous theorem is indeed stronger than Corollary 2.2.14, we note that for $w \in \mathbb{D}$ we have

$$(1-|w|)^{1+\alpha+\max_j(\beta_j-1)_+ +\tau} \geq (1-|w|)^{1+\alpha+\max_j \beta_j + \tau},$$

so the finiteness of the sum in (2.31) implies the finiteness of the sum in (2.25), but not vice versa. This improvement is mainly due to the fact that in estimate (2.33), the radial explosion of h contributes in a different way than the explosion due to one of the singular points.

In the remaining part of this section, let us consider functions $h \in \mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$ with $\gamma \neq 0$. To this end, let us recall that for h in this class we know that $h(0) = 1$ and that

$$\log|h(w)| = O(|w|^\gamma), \quad \text{if } |w| \to 0.$$

We would like to use this behavior to show that, for this class of functions, the sum on the left-hand side of (2.31) can be replaced by

$$\sum_{w\in\mathcal{Z}(h)} \frac{(1-|w|)^{1+\alpha+\max_j(\beta_j-1)_+ +\tau}}{|w|^x} \tag{2.34}$$

for a suitable choice of $x = x(\gamma) > 0$. Obviously, since we always assume that $h(0) = 1$, the sum (2.34) will be finite. However, as we have explained in Remark 2.2.9, it would be nice to obtain an explicit estimate on this sum in terms of K. One way to obtain such an estimate is to apply Proposition 2.3.1 with the function $\varphi(r) = (1-r)^q r^{-x}$, similar to the proof of Theorem 2.3.3. However, since for this choice of φ checking the assumptions of the proposition would require some rather tedious computations, we will choose a different approach.

At first, let us provide an estimate on the counting function $N(h, r)$ for small $r > 0$.

Lemma 2.3.4. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$. Then for $r \in (0, \frac{1}{2}]$ we have*

$$N(h,r) \leq C(\alpha, \vec{\beta}, \vec{\xi}) K r^{\gamma}. \tag{2.35}$$

Proof. Let $0 < r < s < 1$. Then

$$N(h,r) = \frac{1}{\log(\frac{s}{r})} \int_r^s \frac{N(h,r)}{t} dt \leq \frac{1}{\log(\frac{s}{r})} \int_r^s \frac{N(h,t)}{t} dt \leq \frac{1}{\log(\frac{s}{r})} \int_0^s \frac{N(h,t)}{t} dt.$$

Jensen's identity and our assumptions on h thus imply that

$$N(h,r) \quad \leq \quad \frac{1}{\log(\frac{s}{r})} \frac{1}{2\pi} \int_0^{2\pi} \log|h(se^{i\theta})| d\theta \leq \frac{1}{\log(\frac{s}{r})} \frac{K s^{\gamma}}{(1-s)^{\alpha}} \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^N \frac{1}{|se^{i\theta} - \xi_j|^{\beta_j}} d\theta.$$

Choosing $s = \frac{3}{2} r$ (such that $s \leq \frac{3}{4}$) concludes the proof. ∎

The information offered by the previous lemma can immediately be applied to obtain the following result.

Lemma 2.3.5. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$. Then for every $\varepsilon > 0$ we have*

$$\sum_{w\in\mathcal{Z}(h),|w|\leq\frac{1}{2}} \frac{1}{|w|^{(\gamma-\varepsilon)_+}} \leq C(\alpha, \vec{\beta}, \gamma, \vec{\xi}, \varepsilon) K. \tag{2.36}$$

Proof. In view of Lemma 2.3.4 we only need to consider the case $\gamma > \varepsilon$. In this case, we can rewrite the sum in (2.36) as follows:

$$\begin{aligned} \sum_{w\in\mathcal{Z}(h),|w|\leq\frac{1}{2}} \frac{1}{|w|^{\gamma-\varepsilon}} &= (\gamma-\varepsilon) \sum_{w\in\mathcal{Z}(h),|w|\leq\frac{1}{2}} \int_0^{\frac{1}{|w|}} dt\, t^{\gamma-1-\varepsilon} \\ &= (\gamma-\varepsilon) \left[\int_0^2 dt\, t^{\gamma-1-\varepsilon} N(h,1/2) + \int_2^{\infty} dt\, t^{\gamma-1-\varepsilon} N(h,t^{-1}) \right]. \end{aligned}$$

Using Lemma 2.3.4 and the fact that $\gamma > \varepsilon$ we conclude that

$$\int_0^2 dt\, t^{\gamma-1-\varepsilon} N(h,1/2) \;\leq\; C(\alpha,\vec{\beta},\gamma,\vec{\xi},\varepsilon)K.$$

Similarly, using that $\varepsilon > 0$, Lemma 2.3.4 implies that

$$\begin{aligned}\int_2^\infty dt\, t^{\gamma-1-\varepsilon} N(h,t^{-1}) \;&\leq\; C(\alpha,\vec{\beta},\vec{\xi})K \int_2^\infty dt\, t^{-1-\varepsilon} \\ &\leq\; C(\alpha,\vec{\beta},\gamma,\vec{\xi},\varepsilon)K.\end{aligned}$$

This concludes the proof. ■

Remark 2.3.6. We expect that estimate (2.36) remains true if $\varepsilon = 0$. However, we were not able to prove it. □

The next theorem combines the previous lemma with Theorem 2.3.3 to provide the desired bound on the sum in (2.34).

Theorem 2.3.7. *Let $h \in \mathcal{M}(\alpha,\vec{\beta},\gamma,\vec{\xi},K)$. Then for every $\varepsilon,\tau > 0$ we have*

$$\sum_{w\in\mathcal{Z}(h)} \frac{(1-|w|)^{1+\alpha+\max_j(\beta_j-1)_++\tau}}{|w|^{(\gamma-\varepsilon)_+}} \leq C(\alpha,\vec{\beta},\gamma,\vec{\xi},\varepsilon,\tau)K. \tag{2.37}$$

Furthermore, if $\alpha = 0$ and $\max_j \beta_j < 1$, then for every $\varepsilon > 0$ we have

$$\sum_{w\in\mathcal{Z}(h)} \frac{(1-|w|)}{|w|^{(\gamma-\varepsilon)_+}} \leq C(\vec{\beta},\gamma,\vec{\xi},\varepsilon)K. \tag{2.38}$$

Proof. Since the sum on the left-hand side of (2.37) is bounded from above by

$$\sum_{w\in\mathcal{Z}(h),|w|\leq\frac12} \frac{1}{|w|^{(\gamma-\varepsilon)_+}} + C(\gamma,\varepsilon) \sum_{w\in\mathcal{Z}(h),|w|>\frac12} (1-|w|)^{1+\alpha+\max_j(\beta_j-1)_++\tau},$$

we see that the proof of (2.37) is an immediate consequence of estimate (2.31) and Lemma 2.3.5. The proof of (2.38) is analogous, using estimate (2.32) instead of estimate (2.31). ■

2.4. An estimate of Borichev, Golinskii and Kupin

Summary: Improving the results of the previous two sections we provide a non-radial estimate on the zeros of functions in the class $\mathcal{M}(\alpha,\vec{\beta},\gamma,\vec{\xi},K)$.

Broadly speaking, we can summarize the results of the previous two sections as follows: If $h \in \mathcal{M}(\alpha,\vec{\beta},\gamma,\vec{\xi},K)$ then $\sum_{w\in\mathcal{Z}(h)} \varphi(w) \leq C(h,\varphi)$, where $\varphi(w)$ is a suitable function which vanishes on the boundary of the unit disk and which depends on w only through its radial part. To put it differently, these inequalities provide us with uniform estimates on all sequences of zeros, independently of whether these sequences converge to some generic point or to one of the singular points ξ_j. On the other hand, one might suspect that the different behavior of the function at those singular points should also be reflected in the behavior of its sequences of zeros. It is an observation of Borichev, Golinskii and Kupin that this is indeed the case. In (Borichev et al. 2009) they proved the following result.

Theorem 2.4.1. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, 0, \vec{\xi}, K)$, where $\vec{\xi} = (\xi_1, \ldots, \xi_N) \in (\mathbb{T}^N)_*$ and $\vec{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{R}_+^N$. Then for every $\tau > 0$ the following holds: If $\alpha > 0$ then*

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|)^{\alpha+1+\tau} \prod_{j=1}^{N} |w-\xi_j|^{(\beta_j-1+\tau)_+} \leq C(\alpha, \vec{\beta}, \vec{\xi}, \tau) K. \tag{2.39}$$

Furthermore, if $\alpha = 0$ then

$$\sum_{w \in \mathcal{Z}(h)} (1-|w|) \prod_{j=1}^{N} |w-\xi_j|^{(\beta_j-1+\tau)_+} \leq C(\vec{\beta}, \vec{\xi}, \tau) K. \tag{2.40}$$

Remark 2.4.2. It is known that (2.40) can not hold for $\tau = 0$. For instance, Favorov & Golinskii (2009) provided an example of a function $h \in H(\mathbb{D})$, with $h(0) \neq 0$, such that

$$\mathcal{Z}(h) = \{1 - 1/(n+1)\}_{n=1}^{\infty}$$

and $\log|h(w)| \leq K|1-w|^{-1}$ for $w \in \mathbb{D}$. □

Let us note that by applying Lemma 2.3.5, just as we have done in the proof of Theorem 2.3.7, we immediately obtain the following extension of the last theorem.

Theorem 2.4.3. *Let $h \in \mathcal{M}(\alpha, \vec{\beta}, \gamma, \vec{\xi}, K)$, where $\vec{\xi} = (\xi_1, \ldots, \xi_N) \in (\mathbb{T}^N)_*$ and $\vec{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{R}_+^N$. Then for every $\varepsilon, \tau > 0$ the following holds: If $\alpha > 0$ then*

$$\sum_{w \in \mathcal{Z}(h)} \frac{(1-|w|)^{\alpha+1+\tau}}{|w|^{(\gamma-\varepsilon)_+}} \prod_{j=1}^{N} |w-\xi_j|^{(\beta_j-1+\tau)_+} \leq C(\alpha, \vec{\beta}, \gamma, \vec{\xi}, \varepsilon, \tau) K. \tag{2.41}$$

Furthermore, if $\alpha = 0$ then

$$\sum_{w \in \mathcal{Z}(h)} \frac{(1-|w|)}{|w|^{(\gamma-\varepsilon)_+}} \prod_{j=1}^{N} |w-\xi_j|^{(\beta_j-1+\tau)_+} \leq C(\vec{\beta}, \gamma, \vec{\xi}, \varepsilon, \tau) K. \tag{2.42}$$

Remark 2.4.4. Using the inequality

$$(1-|w|)^{\alpha+1+\tau} \prod_{j=1}^{N} |w-\xi_j|^{(\beta_j-1+\tau)_+} \geq C(\vec{\beta}, \vec{\xi}, \tau)(1-|w|)^{\alpha+1+2\tau+\max_j(\beta_j-1)_+}$$

we see that the last theorem is indeed stronger than Theorem 2.3.7. □

The proof of Theorem 2.4.1 that we will sketch below is based on the proof of a generalization of that theorem due to Favorov & Golinskii (2009). These authors realized that Theorem 2.4.1 is a special case of a more general result on the distribution of zeros of subharmonic functions on the unit disk. From our point of view, their proof is simpler and less technical then the original one. On the other hand, it requires some basic facts from the theory of harmonic and subharmonic functions. We refer to Appendix A for a compilation of the necessary background. We will also use the following identity known as the **layer cake representation**, see, e.g., (Lieb & Loss 2001), p.26: For any Borel measure ψ on $\mathbb{C}$ and any ψ-measurable non-negative function f we have

$$\int_{\mathbb{C}} f^p(z)\psi(dz) = p\int_0^\infty t^{p-1}\psi(\{z : f(z) > t\})dt, \quad p > 0. \tag{2.43}$$

Sketch of proof of Theorem 2.4.1. We restrict ourselves to the case of only one singular point, which we assume to be 1, and no radial explosion, i.e., we assume that $h(0) = 1$ and

$$|h(w)| \leq \exp\left(\frac{K}{|w-1|^\beta}\right) \tag{2.44}$$

for some $\beta \geq 1$. So we have to show that

$$\sum_{w\in\mathcal{Z}(h)} (1-|w|)|1-w|^{\beta-1+\tau} \leq C(\beta,\tau)K \tag{2.45}$$

for every $\tau > 0$.

Remark. We remark that the case $|h(w)| \leq \exp\left(K|w-1|^{-\beta}(1-|w|)^{-\alpha}\right)$ can always be reduced to the case (2.44) by considering the sequence of functions $h_n(w) = h((1-2^{-n})w)$, $n \in \mathbb{N}$, which satisfies $|h_n(w)| \leq \exp\left(2^{n\alpha+\beta}K|w-1|^{-\beta}\right)$. Estimate (2.45) then leads to the estimate

$$\sum_{h(w)=0,|w|\leq 1-2^{-n+1}} (1-|w|)|1-w|^{\beta-1+\tau} \leq C(\beta,\tau)\cdot K\cdot 2^{n\alpha},$$

and summing up we obtain

$$\sum_{w\in\mathcal{Z}(h)} (1-|w|)^{\alpha+1+\tau}|1-w|^{\beta-1+\tau} \leq C(\alpha,\beta,\tau)K.$$

□

Let us begin with the proof of (2.45) given assumption (2.44): At first, let μ_h denote the discrete measure supported on $\mathcal{Z}(h)$ such that $\mu_h(\{w\})$ is equal to the order of $w \in \mathcal{Z}(h)$. Setting $\psi(dw) = (1-|w|)\mu_h(dw)$ we see that the sum on the left-hand side of (2.45) is equal to $\int_{\mathbb{D}} |1-w|^{\beta-1+\tau}\psi(dw)$, and we can use (2.43) to derive the identity

$$\sum_{w\in\mathcal{Z}(h)} (1-|w|)|1-w|^{\beta-1+\tau} = C(\beta,\tau)\int_0^2 dt\, t^{\beta-2+\tau}\int_{\Omega_t}(1-|w|)\mu_h(dw),$$

where

$$\Omega_t = \{w \in \mathbb{D} : |w-1| > t\}. \tag{2.46}$$

With a change of variables we can rewrite the last identity as

$$\sum_{w\in\mathcal{Z}(h)}(1-|w|)|1-w|^{\beta-1+\tau}=C(\beta,\tau)\int_0^{\frac{1}{6\pi+3}}dt\, t^{\beta-2+\tau}\int_{\Omega_{2(6\pi+3)t}}(1-|w|)\mu_h(dw). \tag{2.47}$$

The proof of (2.45) now proceeds in two steps. In the first step, which we will skip, it is shown that for $t\in(0,\frac{1}{6\pi+3})$ and $w\in\Omega_{2(6\pi+3)t}$ the following estimate holds:

$$1-|w|\leq 6G_{\Omega_t}(0,w), \tag{2.48}$$

where G_{Ω_t} denotes the Green's function of the set Ω_t, see Appendix A for its definition. The second step consists in proving that for $t\in(0,\frac{1}{(6\pi+3)})$,

$$\int_{\Omega_t}G_{\Omega_t}(0,w)\mu_h(dw)\leq 6K\left(2^{-\beta}+\beta\int_t^2\frac{m(E_s)}{s^{\beta+1}}ds\right), \tag{2.49}$$

where

$$E_s=\{\xi\in\mathbb{T}:|\xi-1|<s\} \tag{2.50}$$

and $m(.)$ denotes normalized Lebesgue measure on $\mathbb{T}$.

Before continuing with the proof of (2.49), let us show that (2.48) and (2.49) indeed imply the validity of (2.45): At first, inequality (2.48) and the inclusion $\Omega_t\supset\Omega_{2(6\pi+3)t}$ imply that

$$\int\limits_{\Omega_{2(6\pi+3)t}}(1-|w|)\mu_h(dw)\leq 6\int\limits_{\Omega_{2(6\pi+3)t}}G_{\Omega_t}(0,w)\mu_h(dw)\leq 6\int\limits_{\Omega_t}G_{\Omega_t}(0,w)\mu_h(dw).$$

Hence, (2.49) shows that the integral on the right-hand side of (2.47) is bounded from above by

$$C(\beta,\tau)K\int_0^{\frac{1}{6\pi+3}}dt\, t^{\beta-2+\tau}\left(2^{-\beta}+\beta\int_t^2\frac{m(E_s)}{s^{\beta+1}}ds\right). \tag{2.51}$$

Since $\beta\geq 1$ and $m(E_s)=O(s)$ for $s\to 0$, we see that the integral in (2.51) is finite and this concludes the proof of (2.45).

Now let us show the validity of (2.49). To begin, we note that $v(w):=\log|h(w)|$ is subharmonic on $\mathbb{D}$, and by assumption we have $v(0)=0$. The Riesz measure $\frac{1}{2\pi}\Delta_v$ coincides with the measure μ_h defined above.

In the following, let $t\in(0,\frac{1}{6\pi+3})$. By assumption (2.44) the function v is bounded above on Ω_t, so, in particular, v has a harmonic majorant on Ω_t. From (A.8), see Appendix A, we thus obtain the representation

$$v(w)=u(w)-\int_{\Omega_t}G_{\Omega_t}(w,z)\mu_h(dz),\quad w\in\Omega_t, \tag{2.52}$$

where u is the least harmonic majorant of v on Ω_t. Setting $w=0$ the last identity shows that

$$\int_{\Omega_t}G_{\Omega_t}(0,z)\mu_h(dz)=u(0). \tag{2.53}$$

Hence, we have reduced the proof of (2.49) to the problem of finding a suitable upper bound on the least harmonic majorant $u(w)$ at $w = 0$.

Let the arc E_t be defined by (2.50). Via the Poisson integral, see (A.2), we can construct a harmonic function $V : \mathbb{D} \to \mathbb{R}$ with boundary values

$$V(\xi) = \begin{cases} t^{-\beta}, & \text{if } \xi \in E_t, \\ |\xi - 1|^{-\beta}, & \text{if } \xi \in \mathbb{T} \setminus E_t. \end{cases} \tag{2.54}$$

Explicitly, V is given by

$$V(w) = \int_{E_t} \frac{1 - |w|^2}{|\xi - w|^2} \frac{m(d\xi)}{t^\beta} + \int_{\mathbb{T}\setminus E_t} \frac{1 - |w|^2}{|\xi - w|^2} \frac{m(d\xi)}{|\xi - 1|^\beta}. \tag{2.55}$$

In particular, we note that V is continuous on $\overline{\mathbb{D}}$.

In the following, let $\omega = \omega_{E_t}$ denote the harmonic measure of E_t with respect to $\mathbb{D}$. Moreover, we set

$$I_t = \{w \in \mathbb{D} : |w - 1| = t\}. \tag{2.56}$$

Using the geometric interpretation of the harmonic measure, see (A.4), we see that $\omega(w) \geq 1/6$ for every $w \in I_t$.

Since $V - t^{-\beta}\omega$ is harmonic on $\mathbb{D}$ and non-negative on $\mathbb{T}$ (with the possible exception of the two boundary points of E_t), Lindelöf's maximum principle implies that $V(w) \geq t^{-\beta}\omega(w)$ for all $w \in \mathbb{D}$. In particular, for $w \in I_t$ we obtain

$$V(w) \geq t^{-\beta}\omega(w) \geq \frac{1}{6t^\beta}. \tag{2.57}$$

Let us recall that $v = \log |h|$ and so by assumption (2.44) we have

$$\limsup_{w\to\xi} v(w) \leq K \cdot \begin{cases} |\xi - 1|^{-\beta}, & \text{if } \xi \in \mathbb{T} \setminus E_t, \\ t^{-\beta}, & \text{if } \xi \in I_t. \end{cases} \tag{2.58}$$

Noting that $\partial\Omega_t = I_t \cup (\mathbb{T} \setminus E_t)$ we thus obtain from (2.54), (2.57) and (2.58) that $\limsup_{w\to\xi} v(w) \leq 6KV(\xi)$ for $\xi \in \partial\Omega_t$. In particular, the maximum principle implies that $v(w) \leq 6KV(w)$ for all $w \in \Omega_t$. Hence, since $w \mapsto 6KV(w)$ is harmonic on Ω_t, it is a majorant of the least harmonic majorant u of v, i.e., we have shown that $u(w) \leq 6KV(w)$ for all $w \in \Omega_t$. In particular, we have

$$u(0) \leq 6KV(0) = 6K \left(\frac{m(E_t)}{t^\beta} + \int_{\mathbb{T}\setminus E_t} \frac{m(d\xi)}{|\xi - 1|^\beta} \right). \tag{2.59}$$

To estimate the integral on the right-hand side of the last equation, we apply the layer cake representation (2.43) and obtain

$$\begin{aligned} \int_{\mathbb{T}\setminus E_t} \frac{m(d\xi)}{|\xi - 1|^\beta} &= \beta \int_0^\infty y^{\beta-1} m(\{\xi \in \mathbb{T} : t < |\xi - 1| < y^{-1}\})\, dy \\ &= 2^{-\beta} - \frac{m(E_t)}{t^\beta} + \beta \int_t^2 \frac{m(E_s)}{s^{\beta+1}}\, ds. \end{aligned} \tag{2.60}$$

The last identity and (2.59) show that

$$u(0) \leq 6K \left(2^{-\beta} + \beta \int_t^2 \frac{m(E_s)}{s^{\beta+1}}\, ds \right), \tag{2.61}$$

which together with (2.53) concludes the proof of (2.49). ■

3. The discrete spectrum of linear operators

In this chapter we will apply the results obtained in Chapter 1 and 2 to derive estimates on the discrete spectrum of linear operators satisfying Assumption 1.5.1. In particular, we will be concerned with the spectrum of non-selfadjoint perturbations of bounded and semibounded selfadjoint operators.

3.1. Some general estimates

Summary: By applying Theorem 2.4.3 to the perturbation determinant of Z by Z_0 we derive estimates on the discrete spectrum of Z.

Throughout this section we assume that Z_0 and Z are operators in $\mathcal{H}$ satisfying Assumption 1.5.1, that is,

(i) Z_0, Z are closed and densely defined with $\rho(Z_0) \cap \rho(Z) \neq \emptyset$,

(ii) $R_Z(b) - R_{Z_0}(b) \in \mathcal{S}_p(\mathcal{H})$ for some $b \in \rho(Z_0) \cap \rho(Z)$ and some fixed $p \in (0, \infty)$,

(iii) $\sigma_d(Z) = \sigma(Z) \cap \rho(Z_0)$.

Moreover, if in addition to (i)-(iii) the operators Z_0 and Z are bounded on $\mathcal{H}$, then we assume that M_1 and M_2 are two fixed bounded operators on $\mathcal{H}$ such that $Z - Z_0 = M_1 M_2$ and

(iv) $M_2 R_{Z_0}(a) M_1 \in \mathcal{S}_p(\mathcal{H})$ for every $a \in \hat{\rho}(Z_0)$.

Remark 3.1.1. Since $Z - Z_0 \in \mathcal{S}_p(\mathcal{H})$ if $Z, Z_0 \in \mathcal{B}(\mathcal{H})$ satisfy assumption (ii), the operator $M_2 R_{Z_0}(a) M_1$ will automatically be in $\mathcal{S}_p(\mathcal{H})$ for the choice $M_1 = I$ and $M_2 = Z - Z_0$. However, for more general choices of M_1 and M_2 assumption (iv) need not be a consequence of assumption (ii). □

If $\Omega \subset \hat{\rho}(Z_0)$ is conformally equivalent to the unit disk, and $a \in \Omega \cap \hat{\rho}(Z)$, then we have already seen in Section 2.1 that the analysis of the discrete spectrum of Z in Ω can be reduced to a study of the holomorphic function $d_a \circ \phi \in H(\mathbb{D})$, where $\phi : \mathbb{D} \to \Omega$ is a conformal mapping which maps 0 onto a, and $d_a = d_a^{Z,Z_0}$ denotes the pth perturbation determinant of Z by Z_0.[1] More precisely, the zero set of $d_a \circ \phi$ coincides with the set

[1] Actually, in Section 2.1 we have only discussed the case when $a = \infty$ and Z, Z_0 are bounded operators on $\mathcal{H}$. However, the general case is completely analogous.

$\phi^{-1}(\sigma_d(Z)\cap\Omega)$ and the multiplicity of $\lambda\in\Omega$ as an eigenvalue of Z is equal to the order of $\phi^{-1}(\lambda)$ as a zero of $d_a\circ\phi$. In this section, by applying Theorem 2.4.3 to the function $d_a\circ\phi$, we will provide estimates on the distribution of the discrete spectrum of Z given the assumption that

$$d_a\circ\phi\in\mathcal{M}(\alpha,\vec{\beta},\gamma,\vec{\xi},K).^2 \tag{3.1}$$

Let us begin by deriving an explicit estimate in terms of Z and Z_0 which guarantees (3.1) to hold. To this end, let $F_a=F_a^{Z,Z_0}:\hat{\rho}(Z_0)\to\mathcal{S}_p(\mathcal{H})$ be defined as follows: If $a\neq\infty$ then we set

$$F_a(\lambda)=[R_Z(a)-R_{Z_0}(a)][(a-\lambda)^{-1}-R_{Z_0}(a)]^{-1}. \tag{3.2}$$

Moreover, if $a=\infty$, which is only possible if $Z_0,Z\in\mathcal{B}(\mathcal{H})$, and if $Z-Z_0=M_1M_2$ with M_1 and M_2 as chosen above, then we define

$$F_\infty(\lambda)=M_2R_{Z_0}(\lambda)M_1. \tag{3.3}$$

Comparing with definitions (1.27), (1.28) and (1.29), respectively, and recalling that $\det_{\lceil p\rceil}(I-(Z-Z_0)R_{Z_0}(\lambda))=\det_{\lceil p\rceil}(I-M_2R_{Z_0}(\lambda)M_1)$ by Proposition 1.4.1, we see that the perturbation determinant d_a can now be expressed as

$$d_a(\lambda)=\det{}_{\lceil p\rceil}(I-F_a(\lambda)),\quad \lambda\in\hat{\rho}(Z_0). \tag{3.4}$$

In addition, the estimates (1.36) and (1.38) show that for $\lambda\in\hat{\rho}(Z_0)$ we have

$$|d_a(\lambda)|\leq\exp\left(\Gamma_p\|F_a(\lambda)\|_{\mathcal{S}_p}^p\right), \tag{3.5}$$

where the constant Γ_p was first defined in Proposition 1.4.3. Hence, for (3.1) to hold, with K replaced with Γ_pK, it is sufficient to assume that

$$\|F_a(\phi(w))\|_{\mathcal{S}_p}^p\leq\frac{K|w|^\gamma}{(1-|w|)^\alpha\prod_{j=1}^N|w-\xi_j|^{\beta_j}},\quad w\in\mathbb{D}.$$

The following theorem is now a direct consequence of Theorem 2.4.3 and our previous discussion.

Theorem 3.1.2. *Let $\Omega\subset\hat{\rho}(Z_0)$ be conformally equivalent to the unit disk, and for $a\in\Omega\cap\hat{\rho}(Z)$ let $\phi:\mathbb{D}\to\Omega$ be a corresponding conformal mapping with $\phi(0)=a$. Let $F_a=F_a^{Z,Z_0}$ be defined by (3.2) and (3.3), respectively, and suppose that for every $w\in\mathbb{D}$ we have*

$$\|F_a(\phi(w))\|_{\mathcal{S}_p}^p\leq\frac{K|w|^\gamma}{(1-|w|)^\alpha\prod_{j=1}^N|w-\xi_j|^{\beta_j}}, \tag{3.6}$$

[2] We note that $(d_a\circ\phi)(0)=d_a(\phi(0))=d_a(a)=1$, so for (3.1) to hold it is sufficient that $d_a\circ\phi$ satisfies estimate (2.9).

where $\alpha, \beta_j, \gamma, K$ are non-negative and $(\xi_1, \ldots, \xi_N) \in (\mathbb{T}^N)_*$. Then for every $\varepsilon, \tau > 0$ the following holds: If $\alpha > 0$ then

$$\sum_{\lambda \in \sigma_d(Z) \cap \Omega} \frac{(1 - |\phi^{-1}(\lambda)|)^{\alpha+1+\tau}}{|\phi^{-1}(\lambda)|^{(\gamma-\varepsilon)_+}} \prod_{j=1}^{N} |\phi^{-1}(\lambda) - \xi_j|^{(\beta_j - 1 + \tau)_+} \leq C(\alpha, \vec{\beta}, \gamma, \vec{\xi}, \varepsilon, \tau, p) K, \quad (3.7)$$

where each eigenvalue is counted according to its multiplicity. Moreover, if $\alpha = 0$ then the same inequality holds with $\alpha + 1 + \tau$ replaced by 1.

Convention 3.1.3. In the following, let us agree that whenever a sum involving eigenvalues is considered each eigenvalue is counted according to its multiplicity. □

Remark 3.1.4. It remains an interesting open question whether (3.7) is still valid when $\tau = 0$ and $\varepsilon = 0$, respectively. At the moment, however, even for the specific choices of Z_0 considered below, we are neither able to answer the corresponding question in the affirmative, nor to provide a suitable counterexample. □

We can obtain a more "explicit" version of the last theorem if Ω is a subset of the plane. To this end, we will need the following distortion theorem, see (Pommerenke 1992), page 9.

Theorem 3.1.5. *Let $\Omega \subset \mathbb{C}$ and let $\Phi : \mathbb{D} \to \Omega$ be conformal. Then*

$$\frac{1}{4}|\Phi'(w)|(1 - |w|) \leq \operatorname{dist}(\Phi(w), \partial\Omega) \leq 2|\Phi'(w)|(1 - |w|) \quad (3.8)$$

for $w \in \mathbb{D}$.

We will refer to the last theorem as the **Koebe distortion theorem**.

Corollary 3.1.6. *Let $\Omega \subset \rho(Z_0)$ be conformally equivalent to the unit disk, and for $a \in \Omega \cap \rho(Z)$ let $\phi : \mathbb{D} \to \Omega$ be a corresponding conformal mapping with $\phi(0) = a$. Let $F_a = F_a^{Z,Z_0}$ be defined by (3.2) and suppose that for every $w \in \mathbb{D}$ we have*

$$\|F_a(\phi(w))\|_{\mathcal{S}_p}^p \leq \frac{K|w|^\gamma}{(1 - |w|)^\alpha \prod_{j=1}^N |w - \xi_j|^{\beta_j}}, \quad (3.9)$$

where $\alpha, \beta_j, \gamma, K$ are non-negative and $(\xi_1, \ldots, \xi_N) \in (\mathbb{T}^N)_*$. Then for every $\varepsilon, \tau > 0$ the following holds: If $\alpha > 0$ then

$$\sum_{\lambda \in \sigma_d(Z) \cap \Omega} \frac{(\operatorname{dist}(\lambda, \partial\Omega)|(\phi^{-1})'(\lambda)|)^{\alpha+1+\tau}}{|\phi^{-1}(\lambda)|^{(\gamma-\varepsilon)_+}} \prod_{j=1}^{N} |\phi^{-1}(\lambda) - \xi_j|^{(\beta_j - 1 + \tau)_+}$$
$$\leq C(\alpha, \vec{\beta}, \gamma, \vec{\xi}, \varepsilon, \tau, p) K. \quad (3.10)$$

Moreover, if $\alpha = 0$ then the same inequality holds with $\alpha + 1 + \tau$ replaced by 1.

Proof. The corollary is a consequence of Theorem 3.1.2 and the fact that, by Koebe's distortion theorem, for $\lambda \in \Omega$ we have

$$4\,\mathrm{dist}(\lambda, \partial\Omega) \geq (1 - |\phi^{-1}(\lambda)|)|\phi'(\phi^{-1}(\lambda))| \geq \frac{1}{2}\,\mathrm{dist}(\lambda, \partial\Omega).$$

■

Let us demonstrate the usefulness of Theorem 3.1.2 by considering a first example.

Example 3.1.7. Let $Z_0 \in \mathcal{B}(\mathcal{H})$ be a normal[3] operator satisfying $\sigma(Z_0) = \sigma_{ess}(Z_0) = \overline{\mathbb{D}}$, and let $Z = Z_0 + M$ where $M \in \mathcal{S}_p(\mathcal{H})$. With the notation from above we then have $Z - Z_0 = M_1 M_2$ where $M_1 = I$ and $M_2 = M$. Moreover, Assumption 1.5.1 is satisfied by Proposition 1.1.9 and we have $\sigma(Z) = \overline{\mathbb{D}} \,\dot{\cup}\, \sigma_d(Z)$. In particular, we can apply Theorem 3.1.2 choosing $\Omega = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $a = \infty$. A conformal map $\phi : \mathbb{D} \to \Omega$, mapping 0 onto ∞, is given by $\phi(w) = w^{-1}$, and we have

$$F_\infty^{Z,Z_0}(\phi(w)) = M_2 R_{Z_0}(w^{-1}) M_1 = M R_{Z_0}(w^{-1}).$$

Using that $\|R_{Z_0}(w^{-1})\| = \mathrm{dist}(w^{-1}, \mathbb{T})^{-1} = |w|(1 - |w|)^{-1}$, we obtain

$$\|F_\infty^{Z,Z_0}(\phi(w))\|_{\mathcal{S}_p}^p \leq \|M\|_{\mathcal{S}_p}^p |w|^p (1 - |w|)^{-p}, \quad w \in \mathbb{D}.$$

Hence, applying Theorem 3.1.2 with $\alpha = \gamma = p$, $\vec{\beta} = \vec{0}$ and $K = \|M\|_{\mathcal{S}_p}^p$, we can conclude that for $\tau \in (0, p)$ (choosing $\varepsilon = \tau$)

$$\sum_{\lambda \in \sigma_d(Z)} \frac{(|\lambda| - 1)^{p+1+\tau}}{|\lambda|^{1+2\tau}} = \sum_{\lambda \in \sigma_d(Z)} \frac{(1 - |\phi^{-1}(\lambda)|)^{p+1+\tau}}{|\phi^{-1}(\lambda)|^{p-\tau}} \leq C(p, \tau)\|M\|_{\mathcal{S}_p}^p.$$

□

In the previous example, Theorem 3.1.2 could be applied almost directly and without further difficulty due to the simple structure of the conformal mapping ϕ. However, for different choices of Z_0 and Ω this mapping will usually take a more complex form and so checking the assumptions of the theorem as well as bringing estimate (3.7) into an explicit form can be considerably more difficult. For this reason, in the following sections we will derive some explicit versions of Theorem 3.1.2 and Corollary 3.1.6 given more specific assumptions on the operator Z_0. More precisely, we will derive estimates on $\sigma_d(Z)$ given that Z_0 is selfadjoint with $\sigma_d(Z_0) = \emptyset$, mainly restricting ourselves to the case when the spectrum of Z_0 is an interval (however, we will shortly discuss the case when the spectrum of Z_0 contains a gap in Section 3.4). It should be clear from the above example that the assumption of selfadjointness of Z_0 is by no means necessary. However, in this thesis we will focus on perturbations of selfadjoint operators.

[3] $Z_0 \in \mathcal{B}(\mathcal{H})$ is normal if $Z_0 Z_0^* = Z_0^* Z_0$. In particular, the spectral theorem for normal operators implies that $\|R_{Z_0}(\lambda)\| = \mathrm{dist}(\lambda, \sigma(Z_0))^{-1}$, $\lambda \in \hat{\rho}(Z_0)$.

3.2. Perturbations of bounded selfadjoint operators

Summary: We apply the results of the previous section to obtain estimates on the discrete spectrum of perturbations of bounded selfadjoint operators.

This section is based on material from the joint work (Hansmann & Katriel 2009).

Throughout this section we assume that $A_0 \in \mathcal{B}(\mathcal{H})$ is selfadjoint with $\sigma(A_0) = [a,b]$, where $a < b$, and that $A = A_0 + M$ where $M \in \mathcal{S}_p(\mathcal{H})$ for some fixed $p \in (0,\infty)$. Moreover, we assume that M_1 and M_2 are two bounded operators on $\mathcal{H}$ such that $M = M_1 M_2$ and

$$M_2 R_{A_0}(\lambda) M_1 \in \mathcal{S}_p(\mathcal{H}) \tag{3.11}$$

for every $\lambda \in \hat{\rho}(A_0)$. In particular, A_0 and A satisfy Assumption 1.5.1 by Remark 1.5.2 (with $Z_0 = A_0$ and $Z = A$, respectively), and we have

$$\sigma(A) = [a,b] \,\dot{\cup}\, \sigma_d(A).$$

Let us define a conformal map $\phi_1 : \mathbb{D} \to \hat{\mathbb{C}} \setminus [a,b]$, mapping 0 onto ∞, by setting

$$\phi_1(w) = \frac{b-a}{4}(w + w^{-1} + 2) + a, \quad w \in \mathbb{D}. \tag{3.12}$$

To adapt Theorem 3.1.2 to the present context we will need the following lemma.

Lemma 3.2.1. *For $w \in \mathbb{D}$ let $\phi_1(w)$ be defined by (3.12). Then*

$$\frac{b-a}{8} \frac{|w^2-1|(1-|w|)}{|w|} \leq \operatorname{dist}(\phi_1(w),[a,b]) \leq \frac{(b-a)(1+\sqrt{2})}{8} \frac{|w^2-1|(1-|w|)}{|w|}.$$

Remark 3.2.2. We note that Koebe's distortion theorem cannot be applied in the derivation of the previous estimate since $\phi_1(0) = \infty$ and so $\phi_1(\mathbb{D})$ is not a subset of the complex plane. □

Proof of Lemma 3.2.1. To begin, we note that

$$\operatorname{dist}(\phi_1(w),[a,b]) = \frac{b-a}{4} \operatorname{dist}(w+w^{-1},[-2,2]). \tag{3.13}$$

For $\lambda = w + w^{-1}, w \in \mathbb{D} \setminus \{0\}$, we define $G_1 = \{w : \operatorname{Re}\lambda \leq -2\}$, $G_2 = \{w : \operatorname{Re}\lambda \geq 2\}$ and $G_3 = \{w : |\operatorname{Re}\lambda| < 2\}$. Then

$$\operatorname{dist}(\lambda,[-2,2]) \;=\; \begin{cases} |\lambda+2| = \frac{|1+w|^2}{|w|}, & \text{if } w \in G_1, \\ |\lambda-2| = \frac{|1-w|^2}{|w|}, & \text{if } w \in G_2, \\ |\operatorname{Im}\lambda| = |\operatorname{Im} w| \frac{1-|w|^2}{|w|^2} & \text{if } w \in G_3. \end{cases} \tag{3.14}$$

We first show that for $w \in G_3$ the following holds:

$$\frac{1}{\sqrt{2}} \frac{|w^2-1|(1-|w|)}{|w|} \leq \operatorname{dist}(\lambda,[-2,2]) \leq \frac{1+\sqrt{2}}{2} \frac{|w^2-1|(1-|w|)}{|w|}. \tag{3.15}$$

With (3.14) this is equivalent to

$$\frac{1}{\sqrt{2}} \leq |\operatorname{Im} w| \frac{1+|w|}{|w||w^2-1|} \leq \frac{1+\sqrt{2}}{2}. \tag{3.16}$$

Switching to polar coordinates we see that $re^{i\theta} \in G_3$ if $\cos^2(\theta) < 4\frac{r^2}{(1+r^2)^2}$ and (3.16) can be rewritten as

$$\frac{1}{\sqrt{2}} \leq \frac{(1+r)\sqrt{1-\cos^2(\theta)}}{\sqrt{(1+r^2)^2 - 4r^2\cos^2(\theta)}} \leq \frac{1+\sqrt{2}}{2}. \tag{3.17}$$

For $x = \cos^2(\theta)$ and fixed r we define

$$f(x) = \frac{1-x}{(1+r^2)^2 - 4r^2 x} \quad, 0 \leq x < 4\frac{r^2}{(1+r^2)^2}.$$

Then f is monotonically decreasing and we obtain

$$\frac{1}{1+6r^2+r^4} = f\left(4\frac{r^2}{(1+r^2)^2}\right) \leq f(x) \leq f(0) = \frac{1}{(1+r^2)^2}.$$

The last chain of inequalities implies the validity of (3.17) (and hence of (3.15)) since

$$\sup_{r\in[0,1]} \frac{1+r}{1+r^2} = \frac{1+\sqrt{2}}{2} \quad \text{and} \quad \inf_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} = \frac{1}{\sqrt{2}}.$$

Next, we show that for $w \in G_1 \cup G_2$ we have

$$\frac{1}{2}\frac{|w^2-1|(1-|w|)}{|w|} \leq \operatorname{dist}(\lambda, [-2,2]) \leq \frac{1+\sqrt{2}}{2}\frac{|w^2-1|(1-|w|)}{|w|}. \tag{3.18}$$

By symmetry, it is sufficient to show it for $w \in G_1$, i.e., for $w \in G_1$

$$\frac{1}{2}\frac{|w^2-1|(1-|w|)}{|w|} \leq \frac{|w+1|^2}{|w|} \leq \frac{1+\sqrt{2}}{2}\frac{|w^2-1|(1-|w|)}{|w|}. \tag{3.19}$$

In polar coordinates this is equivalent to

$$\frac{1}{2} \leq \frac{1}{1-r}\sqrt{\frac{r^2+1+2r\cos(\theta)}{r^2+1-2r\cos(\theta)}} \leq \frac{1+\sqrt{2}}{2} \tag{3.20}$$

for $\cos(\theta) \leq -2\frac{r}{1+r^2}$. For $y = \cos(\theta)$ and fixed r we define

$$q(y) = \frac{r^2+1+2ry}{r^2+1-2ry}, \qquad -1 \leq y \leq -2\frac{r}{1+r^2}. \tag{3.21}$$

A short calculation shows that q is monotonically increasing and we obtain that

$$\left(\frac{1-r}{1+r}\right)^2 = q(-1) \leq q(y) \leq q\left(-2\frac{r}{1+r^2}\right) = \frac{(1-r^2)^2}{1+6r^2+r^4}. \tag{3.22}$$

(3.21) and (3.22) imply the validity of (3.20) and (3.19) (and hence of (3.18)) since

$$\inf_{r\in[0,1]} \frac{1}{1+r} = \frac{1}{2} \quad \text{and} \quad \sup_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} \leq \frac{1+\sqrt{2}}{2}.$$

Combining (3.18), (3.15) and (3.13) concludes the proof of the lemma. ■

In the following we will derive estimates on $\sigma_d(A)$ given the assumption that for every $\lambda \in \mathbb{C} \setminus [a,b]$ we have

$$\|M_2 R_{A_0}(\lambda) M_1\|_{\mathcal{S}_p}^p \leq K \frac{|\lambda - a|^\beta |\lambda - b|^\beta}{\operatorname{dist}(\lambda, [a,b])^\alpha}, \tag{3.23}$$

where $\alpha, K \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ and $\alpha \geq 2\beta$. While one could certainly imagine different assumptions on the norm of $M_2 R_{A_0}(\lambda) M_1$, like a different behavior at the boundary points a and b, the choice above is sufficiently general for the applications we have in mind.

Theorem 3.2.3. *With the assumptions and notations from above, suppose that $M_2 R_{A_0}(\lambda) M_1$ satisfies estimate (3.23) for every $\lambda \in \mathbb{C} \setminus [a,b]$. Let $\tau \in (0,1)$ and define*

$$\begin{aligned} \eta_1 &= \alpha + 1 + \tau, \\ \eta_2 &= (\alpha - 2\beta - 1 + \tau)_+. \end{aligned} \tag{3.24}$$

Then the following holds: If $\alpha > 0$ then

$$\sum_{\lambda \in \sigma_d(A)} \frac{\operatorname{dist}(\lambda, [a,b])^{\eta_1}}{(|b-\lambda||a-\lambda|)^{\frac{\eta_1 - \eta_2}{2}}} \leq C(\alpha, \beta, \tau, p)(b-a)^{\eta_2 - \alpha + 2\beta} K. \tag{3.25}$$

Moreover, if $\alpha = 0$ then the same inequality holds with η_1 replaced by 1.

For a discussion of the consequences of estimate (3.25) on the discrete spectrum of A we refer to Example 2.1.5.

Proof of Theorem 3.2.3. We consider the case $\alpha > 0$ only. As above, let

$$\lambda = \phi_1(w) = \frac{b-a}{4}(w + w^{-1} + 2) + a, \qquad w \in \mathbb{D}.$$

Then a short computation shows that

$$|a - \lambda| = \frac{b-a}{4} \frac{|w+1|^2}{|w|} \quad \text{and} \quad |b - \lambda| = \frac{b-a}{4} \frac{|w-1|^2}{|w|}. \tag{3.26}$$

Using the last two identities and Lemma 3.2.1, assumption (3.23) can be rewritten as

$$\|M_2 R_{A_0}(\lambda) M_1\|_{\mathcal{S}_p}^p \leq \frac{C(\alpha,\beta)K}{(b-a)^{\alpha - 2\beta}} \frac{|w|^{\alpha - 2\beta}}{(1-|w|)^\alpha |w^2 - 1|^{\alpha - 2\beta}}. \tag{3.27}$$

Let $\varepsilon, \tau > 0$ and let η_1, η_2 be defined by (3.24). Then Theorem 3.1.2 (note that $F_\infty^{A,A_0}(\lambda) = M_2 R_{A_0}(\lambda) M_1$) implies that

$$\sum_{\lambda \in \sigma_d(A)} \frac{(1 - |\phi_1^{-1}(\lambda)|)^{\eta_1}}{|\phi_1^{-1}(\lambda)|^{(\alpha - 2\beta - \varepsilon)_+}} |(\phi_1^{-1}(\lambda))^2 - 1|^{\eta_2} \leq \frac{C(\alpha, \beta, \varepsilon, \tau, p)K}{(b-a)^{\alpha - 2\beta}}. \tag{3.28}$$

Restricting τ to the interval $(0,1)$ and setting $\varepsilon = 1-\tau$, the last inequality can be rewritten as

$$\sum_{\lambda \in \sigma_d(A)} \frac{(1-|\phi_1^{-1}(\lambda)|)^{\eta_1}}{|\phi_1^{-1}(\lambda)|^{\eta_2}} |(\phi_1^{-1}(\lambda))^2 - 1|^{\eta_2} \leq \frac{C(\alpha,\beta,\tau,p)K}{(b-a)^{\alpha-2\beta}}. \tag{3.29}$$

By (3.26) we have

$$|(\phi_1^{-1}(\lambda))^2 - 1| = \frac{4}{b-a}|\phi_1^{-1}(\lambda)|(|\lambda-a||\lambda-b|)^{1/2} \tag{3.30}$$

and so Lemma 3.2.1 implies that

$$\begin{aligned} (1-|\phi_1^{-1}(\lambda)|) &\geq \frac{8}{(1+\sqrt{2})(b-a)} \frac{|\phi_1^{-1}(\lambda)| \operatorname{dist}(\lambda,[a,b])}{|(\phi_1^{-1}(\lambda))^2-1|} \\ &= \frac{2}{(1+\sqrt{2})} \frac{\operatorname{dist}(\lambda,[a,b])}{(|\lambda-a||\lambda-b|)^{1/2}}. \end{aligned} \tag{3.31}$$

Inserting (3.31) and (3.30) into (3.29) concludes the proof. ■

Remark 3.2.4. The left- and right-hand sides of (3.31) are actually equivalent (meaning that the same inequality, with another constant, holds in the other direction as well), so no essential information gets lost in this estimate. Without further mentioning, similar remarks apply throughout this chapter whenever an estimate is derived using Koebe's distortion theorem and Lemma 3.2.1, respectively. □

The previous theorem still relies on a quantitative estimate on the $\mathcal{S}_p$-norm of $M_2 R_{A_0}(\lambda) M_1$. However, choosing $M_1 = I$ and $M_2 = M$, and using the bound

$$\|MR_{A_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \|M\|_{\mathcal{S}_p}^p \|R_{A_0}(\lambda)\|^p \leq \frac{\|M\|_{\mathcal{S}_p}^p}{\operatorname{dist}(\lambda,[a,b])^p},$$

we can obtain an estimate given a purely qualitative assumption.

Corollary 3.2.5. *Let $A_0 \in \mathcal{B}(\mathcal{H})$ be selfadjoint with $\sigma(A_0) = [a,b]$ and let $A = A_0 + M$ where $M \in \mathcal{S}_p(\mathcal{H})$. Then for $\tau \in (0,1)$ the following holds: If $p \geq 1-\tau$ then*

$$\sum_{\lambda \in \sigma_d(A)} \frac{\operatorname{dist}(\lambda,[a,b])^{p+1+\tau}}{|b-\lambda||a-\lambda|} \leq C(p,\tau)(b-a)^{-1+\tau}\|M\|_{\mathcal{S}_p}^p. \tag{3.32}$$

Moreover, if $0 < p < 1-\tau$ then

$$\sum_{\lambda \in \sigma_d(A)} \left(\frac{\operatorname{dist}(\lambda,[a,b])}{|b-\lambda|^{1/2}|a-\lambda|^{1/2}} \right)^{p+1+\tau} \leq C(p,\tau)(b-a)^{-p}\|M\|_{\mathcal{S}_p}^p. \tag{3.33}$$

Proof. Apply Theorem 3.2.3 with $M_1 = I$, $M_2 = M$, $K = \|M\|_{\mathcal{S}_p}^p$, $\alpha = p$ and $\beta = 0$. ■

Remark 3.2.6. Estimate (3.32) improves upon an estimate derived by Borichev et al. (2009). Using Theorem 2.4.1 they showed that

$$\sum_{\lambda \in \sigma_d(A)} \frac{\operatorname{dist}(\lambda, [a,b])^{p+1+\tau}}{|b-\lambda||a-\lambda|} \leq C(p,\tau,\|M\|)(b-a)^{-1+\tau}\|M\|_{\mathcal{S}_p}^p, \quad p \geq 1.^4$$

This estimate is of a more qualitative character than estimate (3.32) since the constant on the right-hand side still depends on the norm of the operator M in some unspecified way. By applying Theorem 2.4.3 instead of Theorem 2.4.1 in the derivation of this estimate, we were able to get rid of this dependence. □

For specific operators an application of Theorem 3.2.3 will usually lead to better estimates than an application of Corollary 3.2.5. This will become particularly clear in Chapter 5, where we apply the above results to obtain estimates on the discrete spectrum of Jacobi operators.

Remark 3.2.7. In view of estimate (3.33) we should mention that, for $\gamma < 1$, it is generally not possible to infer the finiteness of the sum

$$\sum_{\lambda \in \sigma_d(A_0+M)} \left(\frac{\operatorname{dist}(\lambda, [a,b])}{|b-\lambda|^{1/2}|a-\lambda|^{1/2}} \right)^\gamma \tag{3.34}$$

from the mere assumption that $M \in \mathcal{S}_p(\mathcal{H})$ for some $p > 0$. Indeed, if A_0 is the free Jacobi operator (see Chapter 5 for its definition), then for every $\gamma < 1$ we can construct a rank one perturbation M such that the sum in (3.34) diverges.[5] For a detailed discussion of this construction we refer to Appendix C. □

3.3. Perturbations of non-negative operators

Summary: The results of Section 3.1 are applied to obtain estimates on the discrete spectrum of perturbations of non-negative selfadjoint operators.

The estimates presented in this section are based on and extend results presented in (Demuth et al. 2009) and (Demuth et al. 2008).

In this section we assume that H_0 is a selfadjoint operator in $\mathcal{H}$ with $\sigma(H_0) = [0,\infty)$, and that $H \in \mathcal{C}(\mathcal{H})$ is densely defined with

$$R_H(u) - R_{H_0}(u) \in \mathcal{S}_p(\mathcal{H}) \tag{3.35}$$

[4] In (Borichev et al. 2009) the authors derived this inequality for Jacobi operators. However, since they did not use specific properties of Jacobi operators in their proof, the inequality remains valid in the more general setting.

[5] This should be compared with Remark 2.2.6.

for some $u \in \rho(H_0) \cap \rho(H)$ (which we assume to be non-empty) and some fixed $p \in (0, \infty)$. In particular, by Remark 1.1.11 and Remark 1.5.2, H_0 and H satisfy Assumption 1.5.1 (with $Z_0 = H_0$ and $Z = H$, respectively) and we have

$$\sigma(H) = [0, \infty) \mathbin{\dot\cup} \sigma_d(H).$$

Let $a \in \mathbb{R}_- \cap \rho(H)$ and choose $b > 0$ such that $a = -b^2$.[6] A conformal mapping ϕ_2 of $\mathbb{D}$ onto $\mathbb{C} \setminus [0, \infty)$, which maps 0 onto a, is given by

$$\phi_2(w) = a \left(\frac{1+w}{1-w} \right)^2, \quad \phi_2^{-1}(\lambda) = \frac{\sqrt{-\lambda} - b}{\sqrt{-\lambda} + b}. \tag{3.36}$$

Here the square root is chosen such that $\mathrm{Re}(\sqrt{-\lambda}) > 0$ for $\lambda \in \mathbb{C} \setminus [0, \infty)$. We note that $\phi_2(-1) = 0$ and $\phi_2(1) = \infty$.

To begin, let us use Corollary 3.2.5 to derive a bound on $\sigma_d(H)$ given the purely qualitative assumption (3.35).

Theorem 3.3.1. *Let H_0 and H be defined as above and assume that $R_H(a) - R_{H_0}(a) \in \mathcal{S}_p(\mathcal{H})$ for some $a \in \mathbb{R}_- \cap \rho(H)$ and some $p \in (0, \infty)$. Then for $\tau \in (0, 1)$ the following holds: If $p \geq 1 - \tau$ then*

$$\sum_{\lambda \in \sigma_d(H)} \frac{\mathrm{dist}(\lambda, [0, \infty))^{p+1+\tau}}{|\lambda||a - \lambda|^{p-1+\tau}(|\lambda| + |a|)^{p+1+\tau}} \leq C(a, p, \tau) \| R_H(a) - R_{H_0}(a) \|_{\mathcal{S}_p}^p. \tag{3.37}$$

Moreover, if $p < 1 - \tau$ then

$$\sum_{\lambda \in \sigma_d(H)} \left(\frac{\mathrm{dist}(\lambda, [0, \infty))}{|\lambda|^{1/2}(|\lambda| + |a|)} \right)^{p+1+\tau} \leq C(a, p, \tau) \| R_H(a) - R_{H_0}(a) \|_{\mathcal{S}_p}^p. \tag{3.38}$$

Remark 3.3.2. A weaker version of the previous theorem, which was derived using Theorem 2.3.7 instead of Theorem 2.4.3, can be found in (Demuth et al. 2008). □

Proof. Let us define $A_0 = R_{H_0}(a)$ and $A = R_H(a)$. Then $A = A_0 + M$ where

$$M = R_H(a) - R_{H_0}(a) \in \mathcal{S}_p(\mathcal{H}).$$

We will consider the case $p \geq 1 - \tau$ first. Since $\sigma(A_0) = [a^{-1}, 0]$ by Proposition 1.1.6, Corollary 3.2.5 shows that for $\tau \in (0, 1)$ we have

$$\sum_{\mu \in \sigma_d(A)} \frac{\mathrm{dist}(\mu, [a^{-1}, 0])^{p+1+\tau}}{|\mu||a^{-1} - \mu|} \leq C(a, p, \tau) \|M\|_{\mathcal{S}_p}^p.$$

We note that $\mu \in \sigma(A)$ if and only if $a - \frac{1}{\mu} \in \sigma(H)$, so the previous inequality can be rewritten as

$$\sum_{\lambda \in \sigma_d(H)} \frac{\mathrm{dist}((a - \lambda)^{-1}, [a^{-1}, 0])^{p+1+\tau} |a - \lambda|^2}{|\lambda|} \leq C(a, p, \tau) \|M\|_{\mathcal{S}_p}^p. \tag{3.39}$$

[6] While the assumption that $a \in (-\infty, 0)$ is not strictly necessary from a technical point of view, it simplifies the following computations considerably.

Similarly, if $p < 1-\tau$ then Corollary 3.2.5 implies that

$$\sum_{\lambda \in \sigma_d(H)} \left(\frac{\operatorname{dist}((a-\lambda)^{-1}, [a^{-1}, 0])|a-\lambda|}{|\lambda|^{1/2}} \right)^{p+1+\tau} \leq C(a,p,\tau)\|M\|_{\mathcal{S}_p}^p. \tag{3.40}$$

Since, as will be shown below, for $\lambda \in \mathbb{C} \setminus [0,\infty)$ we have

$$\operatorname{dist}((a-\lambda)^{-1}, [a^{-1}, 0]) \geq \frac{\operatorname{dist}(\lambda, [0,\infty))}{8|\lambda - a|(|\lambda| + |a|)}, \tag{3.41}$$

(3.39) and (3.40) show the validity of (3.37) and (3.38), respectively.

To show the validity of (3.41) we define (and recall)

$$\phi_1 : \mathbb{D} \to \hat{\mathbb{C}} \setminus [a^{-1}, 0], \quad \phi_1(w) = \frac{1}{4|a|}(w + w^{-1} + 2) + \frac{1}{a}$$

and

$$\phi_2 : \mathbb{D} \to \mathbb{C} \setminus [0,\infty), \quad \phi_2(w) = a\left(\frac{1+w}{1-w}\right)^2.$$

We note that $\phi_1(w) = (a - \phi_2(w))^{-1}$ and $\phi_1(0) = \infty$. Applying Lemma 3.2.1 with $\lambda = \phi_2(w)$ we obtain

$$\operatorname{dist}((a-\lambda)^{-1}, [a^{-1}, 0]) \geq \frac{1}{8|a|} \frac{|(\phi_2^{-1}(\lambda))^2 - 1|(1 - |\phi_2^{-1}(\lambda)|)}{|\phi_2^{-1}(\lambda)|}. \tag{3.42}$$

Koebe's distortion theorem implies that

$$(1 - |\phi_2^{-1}(\lambda)|) \geq \frac{1}{2}|(\phi_2^{-1})'(\lambda)| \operatorname{dist}(\lambda, [0,\infty)).$$

Since $\phi_2^{-1}(\lambda) = \frac{\sqrt{-\lambda}-\sqrt{-a}}{\sqrt{-\lambda}+\sqrt{-a}}$ and $(\phi_2^{-1})'(\lambda) = \frac{-\sqrt{-a}}{\sqrt{-\lambda}(\sqrt{-\lambda}+\sqrt{-a})^2}$, we thus obtain that

$$(1 - |\phi_2^{-1}(\lambda)|) \geq \frac{1}{2} \frac{|a|^{1/2} \operatorname{dist}(\lambda, [0,\infty))}{|\lambda|^{1/2}|\sqrt{-\lambda} + \sqrt{-a}|^2} \geq \frac{1}{4} \frac{|a|^{1/2} \operatorname{dist}(\lambda, [0,\infty))}{|\lambda|^{1/2}(|\lambda| + |a|)}.$$

Using this inequality, and the definition of ϕ_2^{-1}, a short computation shows that (3.42) implies the validity of (3.41). ∎

Remark 3.3.3. Analogous to our discussion in Example 2.1.5, let us consider the consequences of estimate (3.37) on the discrete spectrum of H in a little more detail. To this end, let $\{\lambda_k\}$ be a sequence of isolated eigenvalues of H converging to some $\lambda^* \in [0,\infty)$. Taking a subsequence, we can suppose that one of the following options holds:

(i) $\lambda^* = 0$ and $\operatorname{Re}(\lambda_k) \leq 0$ (ii) $\lambda^* = 0$ and $\operatorname{Re}(\lambda_k) > 0$ (iii) $\lambda^* \in (0,\infty)$.

In case (i), since $\operatorname{dist}(\lambda_k, [0,\infty)) = |\lambda_k|$, (3.37) implies the finiteness of $\sum_k |\lambda_k|^{p+\tau}$, so any such sequence must converge to 0 sufficiently fast. Similarly, in case (ii) estimate (3.37) implies the finiteness of $\sum_k |\operatorname{Im}(\lambda_k)|^{p+1+\tau}|\lambda_k|^{-1}$, and in case (iii) we obtain the finiteness of $\sum_k |\operatorname{Im}(\lambda_k)|^{p+1+\tau}$, which shows that any such sequence must converge to the real line sufficiently fast. Estimate (3.37) also provides information about divergent sequences of eigenvalues. For example, if $\{\lambda_k\}$ is an infinite sequence of eigenvalues which stays bounded away from $[0,\infty)$, that is, $\operatorname{dist}(\lambda_k,[0,\infty)) \geq \delta$ for some $\delta > 0$ and all k, then (3.37) implies that

$$\sum_k \frac{1}{|\lambda_k|^{2p+1+2\tau}} < \infty,$$

which shows that the sequence $\{\lambda_k\}$ must diverge to infinity sufficiently fast. Finally, regarding the consequences of (3.37) on the number of eigenvalues of H, we note that if

$$\Psi_{r,R} = \big\{\lambda \in \mathbb{C} \setminus [0,\infty) : \operatorname{dist}(\lambda,[0,\infty)) \geq r \text{ and } |\lambda| \leq R\big\}$$

where $0 < r < R$, then

$$N(H,\Psi_{r,R}) = \begin{cases} O(R^{2(p+\tau)+1}), & \text{if } r > 0 \text{ is fixed and } R \to \infty, \\ O(r^{-(p+1+\tau)}), & \text{if } R > 0 \text{ is fixed and } r \to 0. \end{cases}$$

□

In the remaining part of this section, to derive an estimate on $\sigma_d(H)$ given a more quantitative assumption, we strengthen our assumptions on H, that is, in addition to the assumptions already stated we assume that $H = H_0 + M$ where M is H_0-compact. We will show in the next lemma that this additional assumption already implies that the spectrum of H is contained in a half-plane. To this end, let us introduce some notation: For $\omega \in \mathbb{R}$ we set

$$\mathbb{H}_\omega^- = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < \omega\} \quad \text{and} \quad \mathbb{H}_\omega^+ = \mathbb{C} \setminus \mathbb{H}_\omega^-, \tag{3.43}$$

and for $Z \in \mathcal{C}(\mathcal{H})$ we define

$$\mathbb{L}(Z) = \left\{\omega \leq 0 : \mathbb{H}_\omega^- \subset \rho(Z) \text{ and } \exists C > 0 \, \forall \lambda \in \mathbb{H}_\omega^- : \|R_Z(\lambda)\| \leq \frac{C}{|\operatorname{Re}(\lambda) - \omega|}\right\}. \tag{3.44}$$

We emphasize that, by definition, $\mathbb{L}(Z)$ is a subset of $\mathbb{R}_-$. Moreover, we note that the constant C in (3.44) may depend on the parameter ω.

Lemma 3.3.4. *Let H_0 be defined as above and let $H = H_0 + M$ where M is H_0-compact. Then $\mathbb{L}(H)$ is non-empty.*

Proof. At first, let us assume that there exists $\omega \leq 0$ such that for every $\lambda \in \mathbb{H}_\omega^-$ we have

$$\|MR_{H_0}(\lambda)\| \leq 1/2. \tag{3.45}$$

Given this assumption, the operator $I - MR_{H_0}(\lambda)$ is invertible for every $\lambda \in \mathbb{H}_\omega^-$ and so the inclusion $\mathbb{H}_\omega^- \subset \rho(H)$ is a consequence of the identity

$$\lambda - H = (I - MR_{H_0}(\lambda))(\lambda - H_0), \quad \lambda \in \mathbb{H}_\omega^-.$$

Moreover, this identity also implies that for every $\lambda \in \mathbb{H}_\omega^-$ we have

$$\|R_H(\lambda)\| \leq \|R_{H_0}(\lambda)\| \|(I - MR_{H_0}(\lambda))^{-1}\| \leq \frac{2}{|\lambda|} \leq \frac{2}{|\operatorname{Re}(\lambda) - \omega|}.$$

In conclusion, we see that $\omega \in \mathbb{L}(H)$.

It remains to show that some ω satisfying (3.45) can indeed be found. To this end, in the following let $\operatorname{Re}(\lambda) < 0$. Since M is H_0-bounded with H_0-bound 0, we have

$$\|Mf\| \leq r\|f\| + \frac{1}{8}\|H_0 f\|, \quad f \in \operatorname{Dom}(H_0), \tag{3.46}$$

for some non-negative constant r. Consequently, for $f \in \mathcal{H}$ we have

$$\begin{aligned} \|MR_{H_0}(\lambda)f\| &\leq r\|R_{H_0}(\lambda)f\| + \frac{1}{8}\|H_0 R_{H_0}(\lambda)f\| \\ &\leq r\|R_{H_0}(\lambda)f\| + \frac{1}{8}\|(H_0 - \lambda)R_{H_0}(\lambda)f\| + \frac{|\lambda|}{8}\|R_{H_0}(\lambda)f\| \\ &\leq \left(\frac{r}{|\lambda|} + \frac{1}{8} + \frac{1}{8}\right)\|f\|, \end{aligned}$$

where in the last inequality we used again that $\|R_{H_0}(\lambda)\| = |\lambda|^{-1}$ if $\operatorname{Re}(\lambda) < 0$. Hence, if $|\lambda| \geq 4r$ then $\|MR_{H_0}(\lambda)\| \leq \frac{1}{2}$. Choosing $\omega = -4r$ concludes the proof. ∎

Remark 3.3.5. Actually, the above proof shows that the spectrum of H in the left half-plane is contained in a ball of radius $|\omega|$. With a little more effort the proof can even be adapted to show that for every $\theta \in (0, \pi/2)$ the entire spectrum of H is contained in a sector $\{\lambda : |\arg(\lambda - \omega')| \leq \theta\}$ for a suitable choice of $\omega' = \omega'(\theta) < 0$. □

Example 3.3.6. If the operator $Z \in \mathcal{C}(\mathcal{H})$ is m-sectorial with vertex $\gamma \leq 0$ and semi-angle $\theta \in [0, \pi/2)$,[7] then $\gamma \in \mathbb{L}(Z)$ and $\|R_Z(\lambda)\| \leq \frac{1}{|\operatorname{Re}(\lambda) - \gamma|}$ for every $\lambda \in \mathbb{H}_\gamma^-$ (here the emphasis should be on the constant 1 in the nominator, which is independent of γ). Such operators will be of importance in our consideration of Schrödinger operators in Chapter 6. We refer to Appendix B for the relevant definitions. □

Let $\omega_0 \geq 0$ and suppose that $-\omega_0 \in \mathbb{L}(H)$, where $H = H_0 + M$ is defined as above. Then the second resolvent identity shows that for $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $a < -\omega_0$ (in particular, $a \in \mathbb{R}_- \cap \rho(H)$),

$$\begin{aligned} F_a^{H,H_0}(\lambda) &= [R_H(a) - R_{H_0}(a)][(a - \lambda)^{-1} - R_{H_0}(a)]^{-1} \\ &= (a - \lambda)R_H(a)MR_{H_0}(\lambda). \end{aligned} \tag{3.47}$$

In particular, the last identity shows that the operator $R_H(a)MR_{H_0}(\lambda)$ is in $\mathcal{S}_p(\mathcal{H})$ if and only if the same is true for the resolvent difference $R_H(a) - R_{H_0}(a)$. In the following,

[7] In particular, this implies that $\sigma(Z) \subset \{\lambda : |\arg(\lambda - \gamma)| \leq \theta\}$. However, this condition is not sufficient for m-sectoriality.

we will derive estimates on $\sigma_d(H)$ given the quantitative assumption that for every $\lambda \in \mathbb{C} \setminus [0,\infty)$ we have

$$\|R_H(a)MR_{H_0}(\lambda)\|_{\mathbf{S}_p}^p \leq \frac{K|\lambda|^\beta}{\operatorname{dist}(\lambda,[0,\infty))^\alpha}, \tag{3.48}$$

where α, K are non-negative and $\beta \in \mathbb{R}$ (here the constant K will usually depend on the parameter a).

Theorem 3.3.7. *With the assumptions and notation from above, let $-\omega_0 \in \mathbb{L}(H)$ and suppose that for some $a < -\omega_0$ and all $\lambda \in \mathbb{C} \setminus [0,\infty)$ the operator $R_H(a)MR_{H_0}(\lambda)$ satisfies assumption (3.48). Let $\varepsilon, \tau > 0$ and define*

$$\begin{array}{rcl} \eta_1 &=& \alpha + 1 + \tau, \\ \eta_2 &=& ((\alpha - 2\beta)_+ - 1 + \tau)_+, \\ \eta_3 &=& ((2p - 3\alpha + 2\beta)_+ - 1 + \tau)_+, \\ \eta_4 &=& (p - \varepsilon)_+. \end{array} \tag{3.49}$$

Then the following holds: If $\alpha > 0$ then

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}}(|\lambda|+|a|)^{\eta_1-\eta_4+\frac{\eta_2+\eta_3}{2}}|\lambda-a|^{\eta_4}} \leq C|a|^{-(\frac{\eta_1+\eta_3}{2}-p+\alpha-\beta)}K, \tag{3.50}$$

where $C = C(\alpha,\beta,p,\varepsilon,\tau)$. Furthermore, if $\alpha = 0$ then the same inequality holds with η_1 replaced by 1.

Remark 3.3.8. The parameter $\frac{\eta_1+\eta_3}{2} - p + \alpha - \beta$ is positive, as a short computation shows. □

Proof of Theorem 3.3.7. We consider the case $\alpha > 0$ only. Let $\lambda = \phi_2(w) = a(\frac{1+w}{1-w})^2$ and note that

$$\phi_2(w) - a = \frac{4aw}{(1-w)^2}.$$

Together with assumption (3.48) the last identity implies that

$$|\phi_2(w)-a|^p\|R_H(a)MR_{H_0}(\phi_2(w))\|_{\mathbf{S}_p}^p \leq \frac{4^p|a|^p|w|^p}{|1-w|^{2p}}\frac{K|\phi_2(w)|^\beta}{\operatorname{dist}(\phi_2(w),[0,\infty))^\alpha}. \tag{3.51}$$

Here the left-hand side is equal to $\|F_a^{H,H_0}(\phi_2(w))\|_{\mathbf{S}_p}^p$ by (3.47). Since $\phi_2'(w) = \frac{4a(1+w)}{(1-w)^3}$, we obtain from Koebe's distortion theorem that

$$\operatorname{dist}(\phi_2(w),[0,\infty)) \geq |a|\frac{|1+w|(1-|w|)}{|1-w|^3}.$$

Using this inequality and the definition of ϕ_2 we see that the right-hand side of (3.51) is bounded from above by

$$\frac{4^pK|a|^{p-\alpha+\beta}|w|^p}{(1-|w|)^\alpha|1+w|^{\alpha-2\beta}|1-w|^{2p-3\alpha+2\beta}}.$$

Applying Corollary 3.1.6 with $\Omega = \mathbb{C} \setminus [0, \infty)$, we thus obtain that for $\varepsilon, \tau > 0$

$$\sum_{\lambda \in \sigma_d(H)} \frac{|\operatorname{dist}(\lambda, [0, \infty))(\phi_2^{-1})'(\lambda)|^{\eta_1}}{|\phi_2^{-1}(\lambda)|^{\eta_4}} |\phi_2^{-1}(\lambda) + 1|^{\eta_2} |\phi_2^{-1}(\lambda) - 1|^{\eta_3} \leq C|a|^{p-\alpha+\beta} K, \tag{3.52}$$

where $C = C(\alpha, \beta, p, \varepsilon, \tau)$. We recall that $\phi_2^{-1}(\lambda) = \frac{\sqrt{-\lambda}-b}{\sqrt{-\lambda}+b}$ where $b = \sqrt{-a}$. Since

$$(\phi_2^{-1})'(\lambda) = \frac{-b}{\sqrt{-\lambda}(\sqrt{-\lambda}+b)^2}$$

and

$$\phi_2^{-1}(\lambda) - 1 = \frac{-2b}{\sqrt{-\lambda}+b}, \quad \phi_2^{-1}(\lambda) + 1 = \frac{2\sqrt{-\lambda}}{\sqrt{-\lambda}+b},$$

estimate (3.52) implies that

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1 - \eta_2}{2}} |\sqrt{-\lambda}+b|^{2\eta_1+\eta_2+\eta_3-\eta_4} |\sqrt{-\lambda}-b|^{\eta_4}} \leq C|a|^{p-\alpha+\beta-\frac{\eta_1+\eta_3}{2}} K.$$

We conclude the proof by noting that

$$|\sqrt{-\lambda} - b| = \frac{|\lambda - a|}{|\sqrt{-\lambda}+b|}$$

and

$$|\sqrt{-\lambda}+b| \leq (|\lambda|^{1/2} + b) \leq 2(|\lambda| + |a|)^{1/2}.$$

■

Estimate (3.50) provides us with a family of inequalities parametrized by $a < -\omega_0$. By considering an average of all these inequalities, that is, by multiplying both sides of (3.50) with an a-dependent weight and integrating with respect to a, it should be possible to extract some more information on $\sigma_d(H)$. Clearly, in this context we have to be aware that the constant K on the right-hand side of (3.50) may still depend on the parameter a. However, we can use the estimate $\|R_H(a)\| \leq C(\omega_0)|a+\omega_0|^{-1}$, valid if $-\omega_0 \in \mathbb{L}(H)$, to get rid of this dependence.

Theorem 3.3.9. *With the assumptions and notation from above, let $-\omega_0 \in \mathbb{L}(H)$ with*

$$\|R_H(\lambda)\| \leq C_0 |\operatorname{Re}(\lambda) + \omega_0|^{-1}, \qquad \operatorname{Re}(\lambda) < -\omega_0, \tag{3.53}$$

and suppose that for all $\lambda \in \mathbb{C} \setminus [0, \infty)$ we have

$$\|MR_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{K|\lambda|^\beta}{\operatorname{dist}(\lambda, [0, \infty))^\alpha}, \tag{3.54}$$

where α, K are non-negative and $\beta \in \mathbb{R}$. Let $\tau > 0$ and define

$$\begin{aligned} \eta_0 &= -\alpha + \beta - \tau, \\ \eta_1 &= \alpha + 1 + \tau, \\ \eta_2 &= ((\alpha - 2\beta)_+ - 1 + \tau)_+. \end{aligned} \tag{3.55}$$

Then the following holds: If $\alpha > 0$ then

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\operatorname{dist}(\lambda, [0,\infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1 - \eta_2}{2}} (|\lambda| + \omega_0)^{\eta_0 + \frac{\eta_1+\eta_2}{2}}} \leq C_0^p C(\alpha, \beta, p, \tau)(1+\omega_0)^{\tau} K \tag{3.56}$$

and

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}^c} \frac{\operatorname{dist}(\lambda, [0,\infty))^{\eta_1}}{|\lambda|^{\beta+1+2\tau}} \leq C_0^p C(\alpha, \beta, p, \tau)(1+\omega_0)^{\frac{\eta_1+\eta_2}{2}+\beta-\alpha} K. \tag{3.57}$$

Furthermore, if $\alpha = 0$ then (3.56) and (3.57) hold with η_1 replaced by 1.

Remark 3.3.10. The choice of the unit disk $\mathbb{D}$ in (3.56) and (3.57) is somewhat arbitrary and it can be replaced with any other disk centered at zero. The point of the theorem is that it provides different estimates on eigenvalues accumulating to 0 and to $(0, \infty)$, respectively. □

Remark 3.3.11. We note that the assumption $MR_{H_0}(\lambda) \in \mathcal{S}_p(\mathcal{H})$ implies that $R_H(a) - R_{H_0}(a) \in \mathcal{S}_p(\mathcal{H})$, but not vice versa. Hence, from a qualitative point of view, the assumptions of the previous theorem are more restrictive than the assumptions of Theorem 3.3.1 and Theorem 3.3.7, respectively. □

Proof. Again, we will only consider the case $\alpha > 0$. As above, let $a = -b^2$ and $a < -\omega_0$. Since

$$\|R_H(a)\| \leq \frac{C_0}{|a + \omega_0|}$$

by assumption (3.53), we see that assumption (3.54) implies that

$$\|R_H(a) M R_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{C_0^p K}{|a+\omega_0|^p} \frac{|\lambda|^{\beta}}{\operatorname{dist}(\lambda, [0,\infty))^{\alpha}}. \tag{3.58}$$

For $\varepsilon, \tau > 0$ let η_j, where $j = 1, \ldots, 4$, be defined by (3.49). Then Theorem 3.3.7 implies that

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}} (|\lambda| + |a|)^{\eta_1 - \eta_4 + \frac{\eta_2+\eta_3}{2}} |\lambda - a|^{\eta_4}} \leq \frac{C_0^p C(\alpha, \beta, p, \varepsilon, \tau) K}{|a|^{\frac{\eta_1+\eta_3}{2} - p + \alpha - \beta} |a+\omega_0|^p}.$$

Setting $\varepsilon = \tau$ and using that $|\lambda - a| \leq (|\lambda| + |a|)$, we see that the last inequality implies that

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}} (|\lambda| + |a|)^{\eta_1 + \frac{\eta_2+\eta_3}{2}}} \leq \frac{C_0^p C(\alpha, \beta, p, \tau) K}{|a|^{\frac{\eta_1+\eta_3}{2} - p + \alpha - \beta} |a+\omega_0|^p}. \tag{3.59}$$

To simplify notation, we set $r = |a|$, $C = C(\alpha, \beta, p, \tau)$,

$$\varphi_1 = \frac{\eta_1 + \eta_3}{2} - p + \alpha - \beta \quad \text{and} \quad \varphi_2 = \eta_1 + \frac{\eta_2 + \eta_3}{2}.$$

Note that $\varphi_1, \varphi_2 > 0$. We can rewrite (3.59) as follows

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{\eta_1} r^{\varphi_1} (r - \omega_0)^{p-1+\tau}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}} (|\lambda| + r)^{\varphi_2} (1+r)^{2\tau}} \leq \frac{C_0^p C K}{(r-\omega_0)^{1-\tau}(1+r)^{2\tau}}. \tag{3.60}$$

Next, we integrate both sides of the last inequality with respect to $r \in (\omega_0, \infty)$. Since $\tau > 0$ we obtain for the right-hand side, substituting $s = \frac{r-\omega_0}{1+\omega_0}$,

$$\int_{\omega_0}^{\infty} \frac{dr}{(r-\omega_0)^{1-\tau}(1+r)^{2\tau}} = \frac{1}{(1+\omega_0)^{\tau}} \int_0^{\infty} \frac{ds}{s^{1-\tau}(1+s)^{2\tau}} \leq \frac{C(\tau)}{(1+\omega_0)^{\tau}}. \tag{3.61}$$

Integrating the left-hand side of (3.60), interchanging sum and integral, it follows that

$$\begin{aligned} &\int_{\omega_0}^{\infty} dr \left(\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\eta_1} r^{\varphi_1} (r-\omega_0)^{p-1+\tau}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}} (|\lambda|+r)^{\varphi_2}(1+r)^{2\tau}} \right) \\ = &\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1-\eta_2}{2}}} \int_{\omega_0}^{\infty} dr \frac{(r-\omega_0)^{p-1+\tau} r^{\varphi_1}}{(|\lambda|+r)^{\varphi_2}(1+r)^{2\tau}}. \end{aligned} \tag{3.62}$$

We note that the finiteness of (3.62) is a consequence of (3.61) and (3.60). Substituting $t = \frac{r-\omega_0}{|\lambda|+\omega_0}$, we obtain for the integral in (3.62):

$$\begin{aligned} &\int_{\omega_0}^{\infty} dr \frac{(r-\omega_0)^{p-1+\tau} r^{\varphi_1}}{(|\lambda|+r)^{\varphi_2}(1+r)^{2\tau}} \\ = &\frac{1}{(|\lambda|+\omega_0)^{\varphi_2-p-\tau}} \int_0^{\infty} dt \frac{t^{p-1+\tau}[(|\lambda|+\omega_0)t+\omega_0]^{\varphi_1}}{(t+1)^{\varphi_2}[(|\lambda|+\omega_0)t+\omega_0+1]^{2\tau}} \\ \geq &\frac{1}{(|\lambda|+\omega_0)^{\varphi_2-\varphi_1-p-\tau}} \int_0^{\infty} dt \frac{t^{p-1+\varphi_1+\tau}}{(t+1)^{\varphi_2}[(|\lambda|+\omega_0)t+\omega_0+1]^{2\tau}} \\ \geq &\frac{C(\alpha,\beta,p,\tau)}{(|\lambda|+\omega_0)^{\varphi_2-\varphi_1-p-\tau}\max(|\lambda|+\omega_0, 1+\omega_0)^{2\tau}}. \end{aligned} \tag{3.63}$$

Since $\varphi_2 - \varphi_1 - p - \tau = \frac{\eta_1+\eta_2}{2} + \eta_0$, the last inequality, together with (3.62), (3.61) and (3.60), shows the validity of (3.56). Similarly, noting that $|\lambda| + \omega_0 \leq |\lambda|(1+\omega_0)$ if $|\lambda| \geq 1$, the same inequalities imply the validity of (3.57) since $\frac{\eta_1-\eta_2}{2} + \varphi_2 - \varphi_1 - p + \tau = \beta + 1 + 2\tau$ and $\varphi_2 - \varphi_1 - p = \frac{\eta_1+\eta_2}{2} + \beta - \alpha$. ■

Remark 3.3.12. The previous theorem will be our main tool in deriving new estimates on the discrete spectrum of non-selfadjoint Schrödinger operators, see Chapter 6. □

Let us conclude this section with a comparison of Theorem 3.3.7 and Theorem 3.3.9 given assumption (3.54). To begin, let us note that both results provide similar estimates on sequences of eigenvalues converging to some point in $(0,\infty)$. However, concerning sequences converging to 0 and ∞, respectively, Theorem 3.3.9 can provide stronger estimates. For instance, if $R > 1$ is sufficiently large, then estimate (3.50) implies that (with η_1, η_3 as defined in (3.49))

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}_R^c} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\eta_1}}{|\lambda|^{\frac{3\eta_1+\eta_3}{2}}} < \infty, \tag{3.64}$$

whereas estimate (3.57) shows that

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}_R^c} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\eta_1}}{|\lambda|^{\beta+1+2\tau}} < \infty. \tag{3.65}$$

Since the exponent $(3\eta_1 + \eta_3)/2$ is strictly larger than $\beta + 1 + 2\tau$, we see that the finiteness of the sum in (3.65) implies the finiteness of the sum in (3.64), but not vice versa. Hence, (3.65) is a stronger conclusion. Similarly, if we can choose $\omega_0 = 0$ then for $\varepsilon > 0$ sufficiently small, (3.56) implies that

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}_\varepsilon} \frac{\operatorname{dist}(\lambda, [0, \infty))^{\eta_1}}{|\lambda|^{\beta+1}} < \infty,$$

whereas (3.50) only allows to conclude that

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}_\varepsilon} \frac{\operatorname{dist}(\lambda, [0, \infty))^{\eta_1}}{|\lambda|^{\frac{\eta_1 - \eta_2}{2}}} < \infty.$$

The last conclusion is weaker than the previous one since the exponent $(\eta_1 - \eta_2)/2$ is never larger than $\beta + 1$ (but it can be smaller).

Remark 3.3.13. Theorem 3.3.9 modifies Theorem 1 in (Demuth et al. 2009), which was formulated in terms of the less explicit assumption that

$$\|MR_{H_0}(\mu^2)\|_{\mathbf{S}_p}^p \leq \frac{K|\mu + i|^\delta}{|\operatorname{Im}(\mu)|^\alpha |\mu|^\nu}, \quad \operatorname{Im}(\mu) > 0,$$

and which provides weaker estimates than Theorem 3.3.9 in case that $\omega_0 = 0$. $\square$

3.4. Further examples

Summary: In the first part of this section we continue our discussion of perturbations of non-negative operators, restricting ourselves to a consideration of the discrete spectrum in the left half-plane. In the second part, we study perturbations of selfadjoint operators with a spectral gap.

While Theorem 3.1.2 and Corollary 3.1.6 allow to study the discrete spectrum of a linear operator Z in any set $\Omega \subset \hat{\rho}(Z_0)$ which is conformally equivalent to the unit disk, in the last two sections we applied these results restricting ourselves to the choice $\Omega = \hat{\rho}(Z_0)$. That is, we derived estimates on the entire discrete spectrum of Z (of course, this choice of Ω was possible only because of the special structure of the operators $Z_0 = A_0$ and $Z_0 = H_0$, respectively). However, it may well happen that, for one reason or another, we are interested only in certain subsets of $\sigma_d(Z)$. For instance, we may be interested in an estimate on the discrete spectrum of Z in Ω', where $\Omega' \subset \hat{\rho}(Z_0)$ is conformally equivalent to the unit disk (e.g., we may know in advance that the discrete spectrum of Z is contained in this set). To obtain such an estimate there exist (at least) two alternative approaches:

(i) We can apply Theorem 3.1.2 with $\Omega = \hat{\rho}(Z_0)$ to obtain an estimate of the form $\sum_{\lambda\in\sigma_d(Z)}(\dots) \leq C$ and then derive an estimate on $\sigma_d(Z)\cap\Omega'$ by restricting the sum, i.e.,

$$\sum_{\lambda\in\sigma_d(Z)\cap\Omega'}(\dots) \leq \sum_{\lambda\in\sigma_d(Z)}(\dots) \leq C.$$

(ii) We can apply Theorem 3.1.2 with $\Omega = \Omega'$.

Naturally, this leads to the question whether one of these approaches will generally provide better estimates on $\sigma_d(Z)\cap\Omega'$ than the other. While we will not discuss this question in general, in the first part of this section we will discuss it for the special case when $Z_0 = H_0$ and Ω' coincides with the left half-plane. As will turn out, in this case either approach may, under certain conditions, provide stronger estimates than the other.

In the second part of this section, to provide at least one example where $\hat{\rho}(Z_0)$ is not conformally equivalent to the unit disk, we will study perturbations of a selfadjoint operator whose spectrum may consist of several disjoint intervals. As we will see, also in this case Theorem 3.1.2 will provide valuable information on the discrete spectrum.

3.4.1. Eigenvalues in a half-plane

Let H_0 and $H = H_0 + M$ (where M is H_0-compact) be defined as in Section 3.3. In particular, we have

$$\sigma(H) = [0,\infty)\,\dot{\cup}\,\sigma_d(H).$$

To obtain an estimate on the discrete spectrum of H in the left half-plane $\mathbb{H}_0^- = \{\lambda\in\mathbb{C} : \operatorname{Re}(\lambda) < 0\}$, we define a conformal mapping $\phi_3 : \mathbb{D}\to\mathbb{H}_0^-$ by setting

$$\phi_3(w) = a\frac{1-w}{1+w}, \qquad \phi_3^{-1}(\lambda) = \frac{a-\lambda}{a+\lambda}. \tag{3.66}$$

As in the previous section, we assume that $a < -\omega_0$ where $-\omega_0\in\mathbb{L}(H)$ and ω_0 is some fixed positive number (if we could choose $\omega_0 = 0$ then by definition of $\mathbb{L}(H)$ the left half-plane would be contained in the resolvent set of H).

Theorem 3.4.1. *Let H and H_0 be defined as above, let $-\omega_0\in\mathbb{L}(H)$ and suppose that for some $a < -\omega_0$, $p > 0$, and every $\lambda\in\mathbb{H}_0^-$ we have*

$$\|R_H(a)MR_{H_0}(\lambda)\|_{\mathbf{S}_p}^p \leq K|\lambda|^{-\delta},$$

where δ and K are non-negative. Let $\tau,\varepsilon > 0$ and define

$$\begin{aligned} \varrho_1 &= (\delta - 1 + \tau)_+, \\ \varrho_2 &= ((p-\delta)_+ - 1 + \tau)_+, \\ \varrho_3 &= (p-\varepsilon)_+. \end{aligned} \tag{3.67}$$

Then the following holds:

$$\sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} \frac{|\operatorname{Re}(\lambda)||\lambda|^{\varrho_1}}{|a-\lambda|^{\varrho_3}|a+\lambda|^{2+\varrho_1+\varrho_2-\varrho_3}} \leq C(\delta,p,\varepsilon,\tau)\frac{K}{|a|^{-p+\delta+1+\varrho_2}}. \tag{3.68}$$

Proof. By assumption we have for $w \in \mathbb{D}$ (compare with (3.47))

$$\begin{aligned} \|F_a^{H,H_0}(\phi_3(w))\|_{\mathcal{S}_p}^p &= |a-\phi_3(w)|^p \|R_H(a)MR_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \\ &\leq \left|\frac{2aw}{1+w}\right|^p K \left|\frac{1+w}{a(1-w)}\right|^\delta = \frac{2^p|a|^{p-\delta}K|w|^p}{|1+w|^{p-\delta}|1-w|^\delta}. \end{aligned} \tag{3.69}$$

Hence, for $\varepsilon, \tau > 0$ we obtain from Corollary 3.1.6 that

$$\sum_{\lambda \in \sigma_d(H)\cap\mathbb{H}_0^-} \frac{\operatorname{dist}(\lambda, \partial\mathbb{H}_0^-)|(\phi_3^{-1})'(\lambda)|}{|\phi_3^{-1}(\lambda)|^{\varrho_3}} |\phi_3^{-1}(\lambda)-1|^{\varrho_1}|\phi_3^{-1}(\lambda)+1|^{\varrho_2} \leq C(\delta,p,\varepsilon,\tau)|a|^{p-\delta}K.$$

Noting that for $\lambda \in \mathbb{H}_0^-$ we have $\operatorname{dist}(\lambda, \partial\mathbb{H}_0^-) = |\operatorname{Re}(\lambda)|$, and calculating $\phi_3^{-1}(\lambda) - 1 = \frac{-2\lambda}{a+\lambda}$, $\phi_3^{-1}(\lambda)+1 = \frac{2a}{a+\lambda}$ and $(\phi_3^{-1})'(\lambda) = \frac{-2a}{(a+\lambda)^2}$, the left-hand side of the last inequality can be simplified to

$$2^{1+\varrho_1+\varrho_2}|a|^{1+\varrho_2} \sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} \frac{|\operatorname{Re}(\lambda)||\lambda|^{\varrho_1}}{|a-\lambda|^{\varrho_3}|a+\lambda|^{2+\varrho_1+\varrho_2-\varrho_3}}. \tag{3.70}$$

This concludes the proof. ■

As we have noted in Remark 3.3.5, given the above assumptions the spectrum of H in the left half-plane will be contained in a ball, so the eigenvalues considered in (3.68) stay bounded and can accumulate at 0 only. Hence, the primary conclusion of the previous theorem is that for every $\tau > 0$ we have

$$\sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} |\operatorname{Re}(\lambda)||\lambda|^{(\delta-1+\tau)_+} < \infty, \tag{3.71}$$

as soon as $\|R_H(a)MR_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq K|\lambda|^{-\delta}$ for some $a < -\omega_0$ and every $\lambda \in \mathbb{H}_0^-$.

In the following, we would like to compare estimate (3.71) with the corresponding one obtained from Theorem 3.3.7 by restricting the sum in (3.50) to the left half-plane. In doing so, we obtain that

$$\sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} |\lambda|^{\frac{\eta_1+\eta_2}{2}} = \sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} |\lambda|^{\frac{1}{2}(\alpha+1+\tau+((\alpha-2\beta)_+-1+\tau)_+)} < \infty, \tag{3.72}$$

whenever

$$\|R_H(a)MR_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{K|\lambda|^\beta}{\operatorname{dist}(\lambda,[0,\infty))^\alpha} \tag{3.73}$$

for every $\lambda \in \mathbb{C}\setminus[0,\infty)$ (here $\alpha \geq 0$ and $\beta \in \mathbb{R}$).

We will compare (3.72) and (3.71) given assumption (3.73). To this end, let us note that for λ in the left-half plane, (3.73) can be written as $\|R_H(a)MR_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq K|\lambda|^{-(\alpha-\beta)}$, so in order to compare (3.72) with (3.71) we have to compare (3.72) with the estimate

$$\sum_{\lambda\in\sigma_d(H)\cap\mathbb{H}_0^-} |\operatorname{Re}(\lambda)||\lambda|^{(\alpha-\beta-1+\tau)_+} < \infty. \tag{3.74}$$

Of course, here we have to assume that $\alpha \geq \beta$. For simplicity, let us also assume that $\tau \in (0,1)$ so $((\alpha - 2\beta)_+ - 1 + \tau)_+ = (\alpha - 2\beta - 1 + \tau)_+$.

The first apparent difference between (3.74) and (3.72) is the occurrence of the real part of λ in (3.74). Since the eigenvalues of H can only accumulate at 0 it would of course be much more desirable if $\mathrm{Re}(\lambda)$ could be replaced with the modulus of λ, so in this respect the sum in (3.74) is not as well adapted to the problem as the sum in (3.72). The reason for this defect is that, as we have seen in the proof of Theorem 3.4.1, the estimates provided by Corollary 3.1.6 always include the factor $\mathrm{dist}(\lambda, \partial\Omega)$ and with our choice of $\Omega = \mathbb{H}_0^-$ this factor no longer coincides with the distance of λ to the spectrum of H.

In the following, to take the occurrence of the real part of λ into account, let us compare (3.74) and (3.72) considering their consequences on eigenvalues converging to 0 in a tangential and non-tangential manner (with respect to the imaginary axis), respectively.

We start with the non-tangential case: Restricting the sums in (3.72) and (3.74) to the set $N = \{\lambda \in \mathbb{H}_0^- : |\mathrm{Re}(\lambda)| \geq K|\lambda|\}$, where $K \in (0,1]$ is arbitrary, the finiteness of these restricted sums is equivalent to the finiteness of

$$\sum_{\lambda \in \sigma_d(H) \cap N} |\lambda|^{\frac{1}{2}(\alpha+1+\tau+(\alpha-2\beta-1+\tau)_+)} \quad \text{and} \quad \sum_{\lambda \in \sigma_d(H) \cap N} |\lambda|^{1+(\alpha-\beta-1+\tau)_+},$$

respectively. Hence, their comparison reduces to a comparison of the exponents

$$X = (\alpha + 1 + (\alpha - 2\beta - 1)_+)/2 \quad \text{and} \quad Y = 1 + (\alpha - \beta - 1)_+,$$

where we ignored the parameter τ since it can be made arbitrarily small anyway. More precisely, estimate (3.74) provides a stronger estimate than (3.72) if and only if $X > Y$.

Remark 3.4.2. Note that $X \in [1/2, \infty)$ while $Y \in [1, \infty)$. □

To begin, it is easily seen that $X = Y$ if $\alpha = 1$. In case that $\alpha \in [0,1)$, a short calculation shows that $X = Y$ if $\beta \leq \alpha - 1$ and $X < Y$ if $\beta > \alpha - 1$, so with growing β estimate (3.72) becomes stronger than estimate (3.74). Finally, if $\alpha > 1$ then $X = Y$ if $\beta \leq \frac{\alpha-1}{2}$ and $X > Y$ if $\beta > \frac{\alpha-1}{2}$. Hence, contrary to the previous case, with growing β estimate (3.74) becomes stronger than estimate (3.72). To summarize, regarding their consequences on eigenvalues converging to 0 in a non-tangential manner, we see that either estimate can, depending on the exponent α in estimate (3.73), be superior to the other (see Figure 3.1).

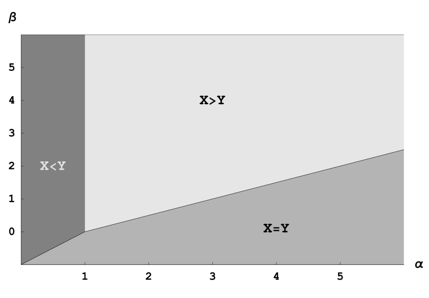

Figure 3.1: The relation between X and Y in the α,β-plane. The half-plane estimate is stronger than the whole-plane estimate if $X > Y$.

Next, let us consider the tangential case. To this end, let us restrict the sums in (3.72) and (3.74) to the set

$$N' = \left\{ \lambda \in \mathbb{H}_0^- : \frac{1}{2} K |\operatorname{Im}(\lambda)|^s \leq |\operatorname{Re}(\lambda)| \leq K |\operatorname{Im}(\lambda)|^s \right\},$$

where $s > 1$ and $K > 0$ are arbitrary. The finiteness of the restricted sums is then equivalent to the finiteness of

$$\sum_{\lambda \in \sigma_d(H) \cap N'} |\lambda|^{\frac{1}{2}(\alpha+1+\tau+(\alpha-2\beta-1+\tau)_+)} \quad \text{and} \quad \sum_{\lambda \in \sigma_d(H) \cap N'} |\lambda|^{s+(\alpha-\beta-1+\tau)_+},$$

and so reduces to a comparison of the exponents

$$X = (\alpha + 1 + (\alpha - 2\beta - 1)_+)/2 \quad \text{and} \quad Y' = s + (\alpha - \beta - 1)_+.$$

In this case, quite contrary to the non-tangential case, a short computation shows that as soon as $s > \max\left(1, \frac{\alpha+1}{2}\right)$ we will have $X < Y'$ independent of the choice of α and β. Consequently, for eigenvalues converging to 0 tangentially (3.72) will generally provide a stronger estimate than (3.74).

In conclusion, while we have seen that both approaches, i.e.,

(i) applying Theorem 3.1.2 with $\Omega = \mathbb{C} \setminus [0, \infty)$ and then restricting to the left half-plane, or

(ii) applying the theorem directly with $\Omega = \mathbb{H}_0^-$,

have certain advantages and disadvantages when considering sequences of eigenvalues converging to 0 non-tangentially, it is generally the first approach that leads to better estimates on sequences converging in a tangential manner.

Remark 3.4.3. By using Corollary 3.1.6 in the proof of Theorem 3.4.1, we have implicitly applied Theorem 2.4.3 to the holomorphic function $h = d_a^{H,H_0} \circ \phi_3$. However, we have not taken into account that in this case the function h is not only holomorphic on the unit disk, but can actually be extended to a holomorphic function on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ (note that $\phi_3^{-1}(\mathbb{R}_+ \setminus \{-a\}) = (-\infty, -1] \cup [1, \infty)$). More precisely, the function h satisfies

$$|h(w)| \leq \exp\left(\frac{K|w|^\gamma}{|w-1|^{\beta_1}|w+1|^{\beta_2}} \right), \quad w \in \mathbb{D},$$

for some $\gamma, \beta_1, \beta_2 > 0$, see (3.69), and for every $s > 1$ it can be extended to an analytic function on

$$\mathbb{D}_s \setminus ([-s, -1] \cup [1, s]) .$$

In particular, the zeros of h in $\mathbb{D}$ (which correspond to the eigenvalues of H in the left half-plane) can accumulate to -1 and 1 only.

It seems to be a reasonable conjecture that for this particular class of functions Theorem 2.4.3 can be improved. For instance, in estimate (2.42) it should be possible to replace the factor $1 - |w|$ with the distance of w to the set $\{-1, 1\}$. In particular, as desired, this would also allow to replace the real part of λ with the modulus of λ in estimate (3.74). $\square$

3.4.2. Spectral gaps

Let D_0 be a selfadjoint operator in $\mathcal{H}$ with $\sigma_{ess}(D_0) = \sigma(D_0)$ and let us assume that the spectrum of D_0 contains at least one gap, that is, there exist $\zeta_1, \zeta_2 \in \sigma(D_0)$, with $\zeta_1 < \zeta_2$, such that the interval (ζ_1, ζ_2) is contained in the resolvent set of D_0. For instance, the spectrum of D_0 may consist of several disjoint intervals, two of which may be unbounded. In particular, we neither assume D_0 to be bounded, nor to be semibounded.

Example 3.4.4. For instance, we can think of D_0 as the free Dirac operator, i.e., $\sigma(D_0) = \mathbb{R} \setminus [-1, 1]$, or as a Schrödinger operator with a periodic potential, i.e., $\sigma(D_0) = \dot{\bigcup}_{n=1}^{\infty} [a_n, a_{n+1}]$, where the sequence $\{a_n\}_{n \in \mathbb{N}}$ is strictly increasing.[8] □

In the following, let $D = D_0 + M$ where M is D_0-compact. Then we have $\sigma(D) = \sigma(D_0) \,\dot{\cup}\, \sigma_d(D)$ by Remark 1.2.9, and so the spectrum of D consists of $\sigma(D_0)$ and an at most countable sequence of eigenvalues of finite type which can accumulate on $\sigma(D_0)$ only. In this case, the (extended) resolvent set of D_0 need not be conformally equivalent to the unit disk and so Theorem 3.1.2 cannot be applied with the choice $\Omega = \hat{\rho}(D_0)$. Hence, in order to apply that theorem we have to restrict ourselves to a consideration of parts of the discrete spectrum of D.

Remark 3.4.5. We will see in the next chapter that for H as chosen in Section 3.3, the estimates derived on $\sigma_d(H)$ in Theorem 3.3.1 can be improved in case that H is selfadjoint. These improvements will be based on the variational characterization of the eigenvalues (of selfadjoint operators) lying below and above the essential spectrum, respectively. While there exist variational characterizations of eigenvalues in gaps of the essential spectrum as well (see, e.g., (Griesemer & Siedentop 1999)), in general these results seem to be more technical and less easy to apply than their semibounded counterparts. For this reason, we think that the results discussed below might also be of relevance in the purely selfadjoint setting. □

Let $\zeta = \frac{1}{2}(\zeta_2 + \zeta_1)$ and $r = \frac{1}{2}(\zeta_2 - \zeta_1)$. In the following, we will derive estimates on the isolated eigenvalues of D in the disk $\mathbb{D}_r(\zeta) = \{z : |z - \zeta| < r\}$, situated in between ζ_1 and ζ_2, and on their rate of convergence to ζ_1 and ζ_2, respectively.

Remark 3.4.6. As we have seen in the previous section, in order to derive estimates on $\sigma_d(D) \cap \mathbb{D}_r(\zeta)$ it can be advantageous to first apply Theorem 3.1.2 to the set $\Omega = \hat{\mathbb{C}} \setminus [\zeta_1, \zeta_2]$ and then to restrict the obtained sum to the disk $\mathbb{D}_r(\zeta)$. However, to simplify matters we will directly apply Theorem 3.1.2 to the set $\Omega = \mathbb{D}_r(\zeta)$. □

For simplicity, let us assume that $\zeta_1 = -1$ and $\zeta_2 = 1$, i.e., $\mathbb{D}_r(\zeta) = \mathbb{D}_1(0) = \mathbb{D}$. This is no great loss of generality since the general case can always be reduced to this case by considering the pair $(f(D_0), f(D))$ where $f(\lambda) = (\lambda - \zeta_1)\frac{2}{\zeta_2 - \zeta_1} - 1$; however, it simplifies computations considerably.

[8] For a precise definition and discussion of Dirac operators and of Schrödinger operators with periodic potentials we refer to (Thaller 1992) and (Reed & Simon 1978), Chapter XIII.16, respectively.

For $a \in [0,1)$ a conformal map $\phi_4 : \mathbb{D} \to \mathbb{D}$, mapping 0 onto ia, is given by

$$\phi_4(w) = \frac{ia - w}{1 + iaw}, \quad w \in \mathbb{D}. \tag{3.75}$$

We note that $\phi_4^{-1}(w) = \phi_4(w)$. In the following, let us set $\nu = \frac{ia-1}{ia+1}$, i.e., $\phi_4(1) = \nu$ and $\phi_4(-1) = -\nu^{-1}$.

Lemma 3.4.7. *Let ϕ_4 and ν be defined as above. If $\lambda = \phi_4(w)$ then*

$$\frac{1-a}{1+a}|w - \nu| \leq |\lambda - 1| \leq \frac{1+a}{1-a}|w - \nu|, \tag{3.76}$$

$$\frac{1-a}{1+a}|w + \nu^{-1}| \leq |\lambda + 1| \leq \frac{1+a}{1-a}|w + \nu^{-1}|, \tag{3.77}$$

$$(1-a)|w| \leq |ia - \phi_4(w)| \leq (1+a)|w| \tag{3.78}$$

and

$$|\phi_4'(w)| \geq \frac{1-a}{1+a}. \tag{3.79}$$

Proof. A direct computation shows that

$$|\lambda - 1| = \left| \frac{1 + ia}{1 + iaw} \right| |\nu - w|.$$

Since $a \in [0,1)$ and $w \in \mathbb{D}$, we have

$$1 - a \leq |1 + iaw| \leq 1 + a, \tag{3.80}$$

and the same inequality holds when $1 + iaw$ is replaced by $1 + ia$. This shows the validity of (3.76). The validity of (3.77) is shown in a similar fashion. Since

$$ia - \phi_4(w) = w\frac{(1+a)(1-a)}{1+iaw} \quad \text{and} \quad \phi_4'(w) = \frac{a^2 - 1}{(1+iaw)^2},$$

(3.80) also shows the validity of (3.78) and (3.79), respectively. ■

Theorem 3.4.8. *Let D_0 be a selfadjoint operator in $\mathcal{H}$ with $(-1,1) \subset \rho(D_0)$ and let $D = D_0 + M$ where M is D_0-compact. Let $ia \in \rho(D)$, where $a \in [0,1)$, and suppose that for $p > 0$ and every $\lambda \in \mathbb{D}$ we have*

$$\|R_D(ia) M R_{D_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{K}{|\lambda^2 - 1|^\beta}, \tag{3.81}$$

where β and K are non-negative. If $\varepsilon, \tau > 0$ and

$$\begin{aligned} \kappa_1 &= (\beta - 1 + \tau)_+, \\ \kappa_2 &= (p - \varepsilon)_+, \\ \kappa_3 &= 2(\beta + \kappa_1) + 1, \end{aligned} \tag{3.82}$$

then the following holds:

$$\sum_{\lambda \in \sigma_d(D) \cap \mathbb{D}} (1 - |\lambda|)|\lambda^2 - 1|^{\kappa_1} \left| \frac{1 + ia\lambda}{ia - \lambda} \right|^{\kappa_2} \leq C(\beta, p, \varepsilon, \tau) \frac{(1+a)^{\kappa_3 + p}}{(1-a)^{\kappa_3}} K.$$

Remark 3.4.9. In view of the discussion made in Remark 3.4.3 above, we conjecture that in the previous estimate the factor $(1-|\lambda|)$ can be replaced with the distance of λ to the set $\{-1,1\}$. □

Of course, a similar estimate can be derived assuming that

$$\|R_D(ia)MR_{D_0}(\lambda)\|_{\mathbf{S}_p}^p \leq K|\lambda-1|^{-\beta_1}|\lambda+1|^{-\beta_2}$$

where $\beta_1 \neq \beta_2$.

Before turning to the proof of Theorem 3.4.8, let us state the following corollary which is a consequence of the fact that $\left|\frac{1+ia\lambda}{ia-\lambda}\right| = |\phi_4(\lambda)|^{-1} > 1$ for every $\lambda \in \mathbb{D}$.

Corollary 3.4.10. *With the assumptions and notation from above, suppose that for $p > 0$ and every $\lambda \in \mathbb{D}$ we have*

$$\|R_D(ia)MR_{D_0}(\lambda)\|_{\mathbf{S}_p}^p \leq \frac{K}{|\lambda^2-1|^{\beta}},$$

where β and K are non-negative. Then for $\tau > 0$ the following holds:

$$\sum_{\lambda\in\sigma_d(D)\cap\mathbb{D}} (1-|\lambda|)|\lambda^2-1|^{(\beta-1+\tau)_+} \leq C(a,\beta,p,\tau)K. \tag{3.83}$$

Remark 3.4.11. If D is a selfadjoint operator then the left-hand side of (3.83) can be estimated from below by

$$\sum_{\lambda\in\sigma_d(D),\lambda<0} |\lambda+1|^{1+(\beta-1+\tau)_+} + \sum_{\lambda\in\sigma_d(D),\lambda>0} |\lambda-1|^{1+(\beta-1+\tau)_+}.$$

□

Proof of Theorem 3.4.8. By Lemma 3.4.7 and assumption (3.81) we have (compare with (3.47))

$$\begin{aligned}\|F_{ia}^{D,D_0}(\phi_4(w))\|_{\mathbf{S}_p}^p &= |\phi_4(w)-ia|^p\|R_D(ia)MR_{D_0}(\lambda)\|_{\mathbf{S}_p}^p \\ &\leq \left(\frac{1+a}{1-a}\right)^{2\beta}\frac{(1+a)^pK|w|^p}{|w-\nu|^{\beta}|w+\nu^{-1}|^{\beta}}.\end{aligned}$$

Hence, Corollary 3.1.6 implies that

$$\begin{aligned}&\sum_{\lambda\in\sigma_d(D)\cap\mathbb{D}} \frac{(1-|\lambda|)|(\phi_4^{-1})'(\lambda)|}{|\phi_4^{-1}(\lambda)|^{\kappa_2}}|\phi_4^{-1}(\lambda)-\nu|^{\kappa_1}|\phi_4^{-1}(\lambda)+\nu^{-1}|^{\kappa_1} \\ \leq\ & C(\beta,p,\varepsilon,\tau)\frac{(1+a)^{2\beta+p}}{(1-a)^{2\beta}}K.\end{aligned} \tag{3.84}$$

Since $\phi_4 = \phi_4^{-1}$, Lemma 3.4.7 shows that the left-hand side of (3.84) is bounded from below by

$$\left(\frac{1-a}{1+a}\right)^{2\kappa_1+1}\sum_{\lambda\in\sigma_d(D)\cap\mathbb{D}}(1-|\lambda|)|\lambda^2-1|^{\kappa_1}\left|\frac{1+ia\lambda}{ia-\lambda}\right|^{\kappa_2}.$$

This concludes the proof. ■

4. A glimpse at selfadjoint operators

In this chapter we will show that for selfadjoint operators (meaning that both the "free" as well as the "perturbed" operator are selfadjoint), using the variational characterization of the discrete spectrum, the estimates established in Corollary 3.2.5 and Theorem 3.3.1 can be improved considerably.

To begin, let us state the following min-max principle for the eigenvalues of selfadjoint operators situated below and above the essential spectrum, respectively. Here $\langle . \, , . \rangle$ denotes the inner product on $\mathcal{H}$, which we assume to be linear in the first and semilinear in the second component.

Proposition 4.1.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint and let*

$$\lambda_1^- \leq \lambda_2^- \leq \ldots \leq \inf \sigma_{ess}(A) =: a \quad \text{and} \quad \lambda_1^+ \geq \lambda_2^+ \geq \ldots \geq \sup \sigma_{ess}(A) =: b$$

denote its eigenvalues of finite type (counted according to multiplicity) situated below and above the essential spectrum, respectively. We set $\lambda_{N+1}^- = \lambda_{N+2}^- = \ldots = a$ ($\lambda_{N+1}^+ = \lambda_{N+2}^+ = \ldots = b$) if there exist only N eigenvalues below a (above b). Then for every $n \in \mathbb{N}$ we have

$$\lambda_n^+ = \inf_{\mathcal{W} \subset \mathcal{H}, \, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle A\psi, \psi \rangle \tag{4.1}$$

and

$$\lambda_n^- = \sup_{\mathcal{W} \subset \mathcal{H}, \, \dim(\mathcal{W}) = n-1} \; \inf_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle A\psi, \psi \rangle. \tag{4.2}$$

For a proof we refer to (Reed & Simon 1978), p.76-78.

The previous proposition allows to derive a variational characterization of the singular values of a compact operator as well.

Proposition 4.1.2. *Let $K \in \mathcal{S}_\infty(\mathcal{H})$ and let $s_1(K) \geq s_2(K) \geq \ldots > 0$ denote its singular values. Then for $n \in \mathbb{N}$ we have*

$$s_n(K) = \inf_{\mathcal{W} \subset \mathcal{H}, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \|K\psi\|. \tag{4.3}$$

Proof. Apply Proposition 4.1.1 to the non-negative compact operator K^*K and compare with Remark 1.3.2. ■

Although the next lemma is an easy consequence of the previous two propositions, it will be our main tool in deriving improved selfadjoint versions of Corollary 3.2.5 and Theorem 3.3.1, respectively.

Lemma 4.1.3. *Let $A_0, A \in \mathcal{B}(\mathcal{H})$ be selfadjoint, with $\sigma(A_0) = \sigma_{ess}(A_0)$, and suppose that $A - A_0 \in \mathcal{S}_\infty(\mathcal{H})$. Let $a = \inf \sigma(A_0)$ and $b = \sup \sigma(A_0)$,*[1] *and, with the same conventions as in Proposition 4.1.1, let $\lambda_n^-(A) \leq a$ and $\lambda_n^+(A) \geq b$ denote the eigenvalues of finite type of A situated below a and above b, respectively. Then for $n \in \mathbb{N}$ we have*

$$a - \lambda_n^-(A) \leq s_n((A - A_0)_-) \tag{4.4}$$

and

$$\lambda_n^+(A) - b \leq s_n((A - A_0)_+). \tag{4.5}$$

Remark 4.1.4. If Z is a selfadjoint operator in $\mathcal{H}$ then $Z_+ = \frac{1}{2}(|Z| + Z)$ and $Z_- = \frac{1}{2}(|Z| - Z)$ denote its positive and negative part, respectively (here $|Z|$ can be defined via the spectral theorem). We note that Z_+ and Z_- are non-negative operators satisfying $Z = Z_+ - Z_-$ and $|Z| = Z_+ + Z_-$. In particular, Z is a compact operator on $\mathcal{H}$ if and only if the same is true of both its positive and negative part. □

Proof of Lemma 4.1.3. Since $A_0 \geq a$ and $(A_0 - A)_+ \geq A_0 - A$, we obtain from Proposition 4.1.1 that

$$\begin{aligned}
a - \lambda_n^-(A) &= a - \sup_{\mathcal{W} \subset \mathcal{H},\, \dim(\mathcal{W}) = n-1} \; \inf_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle A\psi, \psi \rangle \\
&= a + \inf_{\mathcal{W} \subset \mathcal{H},\, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle -A\psi, \psi \rangle \\
&= \inf_{\mathcal{W} \subset \mathcal{H},\, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle (a - A)\psi, \psi \rangle \\
&\leq \inf_{\mathcal{W} \subset \mathcal{H},\, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle (A_0 - A)\psi, \psi \rangle \\
&\leq \inf_{\mathcal{W} \subset \mathcal{H},\, \dim(\mathcal{W}) = n-1} \; \sup_{\psi \in \mathcal{W}^\perp, \|\psi\| = 1} \langle (A_0 - A)_+\psi, \psi \rangle = s_n((A - A_0)_-).
\end{aligned}$$

In the last identity we used that the eigenvalues and singular values of the non-negative compact operator $(A - A_0)_- = (A_0 - A)_+$ coincide. The proof of (4.5) is completely analogous and is therefore omitted. ■

The following theorem provides the desired improvement of Corollary 3.2.5 in the selfadjoint setting.

Theorem 4.1.5. *Let $A_0, A \in \mathcal{B}(\mathcal{H})$ be selfadjoint, with $\sigma(A_0) = \sigma_{ess}(A_0)$, and suppose that $A - A_0 \in \mathcal{S}_p(\mathcal{H})$ for some $p > 0$. If $a = \inf \sigma(A_0)$ and $b = \sup \sigma(A_0)$ then*

$$\sum_{\lambda \in \sigma_d(A), \lambda < a} (a - \lambda)^p \leq \|(A - A_0)_-\|_{\mathcal{S}_p}^p \tag{4.6}$$

and

$$\sum_{\lambda \in \sigma_d(A), \lambda > b} (\lambda - b)^p \leq \|(A - A_0)_+\|_{\mathcal{S}_p}^p. \tag{4.7}$$

In particular,

$$\sum_{\lambda \in \sigma_d(A)} \operatorname{dist}(\lambda, [a, b])^p \leq \|A - A_0\|_{\mathcal{S}_p}^p. \tag{4.8}$$

[1] We note that $a = \inf \sigma_{ess}(A)$ and $b = \sup \sigma_{ess}(A)$ as a consequence of Weyl's theorem.

Proof. The inequalities (4.6) and (4.7) are direct consequences of Lemma 4.1.3. Moreover, (4.8) is a consequence of the fact that $\|A - A_0\|_{\mathcal{S}_p}^p = \|(A - A_0)_+\|_{\mathcal{S}_p}^p + \|(A - A_0)_-\|_{\mathcal{S}_p}^p$. ∎

Remark 4.1.6. We would like to point out that the constant 1 on the right-hand side of (4.6) and (4.7) is optimal, i.e., the inequality

$$\sum_{\lambda \in \sigma_d(A), \lambda > b} (\lambda - b)^p \leq c \, \|(A - A_0)_+\|_{\mathcal{S}_p}^p$$

can generally not hold for any constant $c < 1$. For instance, this can be seen by considering the case where $A_0 = 0$ and $A \in \mathcal{S}_p(\mathcal{H})$ is non-negative. □

In the following, assuming that $\sigma(A_0) = [a, b]$, let us compare (4.8) with the corresponding estimates established in Corollary 3.2.5. There we have shown that for every $\tau \in (0, 1)$ the following holds: If $p \geq 1 - \tau$ then

$$(b - a)^{1-\tau} \sum_{\lambda \in \sigma_d(A)} \frac{\operatorname{dist}(\lambda, [a, b])^{p+1+\tau}}{|b - \lambda||a - \lambda|} \leq C(p, \tau) \|A - A_0\|_{\mathcal{S}_p}^p, \tag{4.9}$$

and if $p \in (0, 1 - \tau)$ then

$$(b - a)^p \sum_{\lambda \in \sigma_d(A)} \left(\frac{\operatorname{dist}(\lambda, [a, b])}{|b - \lambda|^{1/2}|a - \lambda|^{1/2}} \right)^{p+1+\tau} \leq C(p, \tau) \|A - A_0\|_{\mathcal{S}_p}^p. \tag{4.10}$$

A short computation shows that for $\lambda \in \mathbb{R} \setminus [a, b]$ and $\tau \in [0, 1)$

$$\operatorname{dist}(\lambda, [a, b])^p > \begin{cases} (b - a)^{1-\tau} \frac{\operatorname{dist}(\lambda, [a,b])^{p+1+\tau}}{|b-\lambda||a-\lambda|}, & \text{if } p \geq 1 - \tau, \\ (b - a)^p \left(\frac{\operatorname{dist}(\lambda, [a,b])}{|b-\lambda|^{1/2}|a-\lambda|^{1/2}} \right)^{p+1+\tau}, & \text{if } p \in (0, 1 - \tau). \end{cases}$$

Hence, (4.8) is a stronger estimate than (4.9) and (4.10), respectively, even if it would be allowed to choose $\tau = 0$.

From a qualitative point of view, the difference between (4.8) and (4.10) is particularly striking: If $A - A_0 \in \mathcal{S}_p(\mathcal{H})$ for some $p \in (0, 1)$, then (4.8) allows to conclude the finiteness of

$$\sum_{\lambda \in \sigma_d(A), \lambda < a} (a - \lambda)^p, \tag{4.11}$$

whereas (4.10) (setting $\tau = 0$) only implies the finiteness of

$$\sum_{\lambda \in \sigma_d(A), \lambda < a} (a - \lambda)^{\frac{1}{2}(p+1)}. \tag{4.12}$$

In particular, while the exponent in (4.11) can be made arbitrarily small with a suitable choice of p, the exponent in (4.12) will always be larger than 1/2. This shows a real difference in the behavior of the discrete spectrum of A in the selfadjoint and non-selfadjoint cases, since, as we have already discussed in Remark 3.2.7 above, for non-selfadjoint A the finiteness of the sum

$$\sum_{\lambda \in \sigma_d(A), \lambda < a} (a - \lambda)^\gamma,$$

where $\gamma < 1/2$, can generally not be inferred from the mere assumption that $A \in \mathcal{S}_p(\mathcal{H})$ for some $p > 0$ (see Corollary C.3 in Appendix C).

In the remaining part of this chapter, we will apply Theorem 4.1.5 to derive some inequalities on the discrete spectrum of semibounded selfadjoint operators in terms of resolvent and semigroup differences, respectively. We begin with a look at the resolvent case.

Corollary 4.1.7. *Let H_0 and H be selfadjoint operators in $\mathcal{H}$. In addition, assume that $\sigma(H_0) = [0, \infty)$, that H is bounded from below and that $R_H(a) - R_{H_0}(a) \in \mathcal{S}_p(\mathcal{H})$ for some $p > 0$ and some $a < \inf \sigma(H)$. Then*

$$\sum_{\lambda \in \sigma_d(H)} \frac{|\lambda|^p}{|a - \lambda|^p} \leq |a|^p \|(R_H(a) - R_{H_0}(a))_-\|_{\mathcal{S}_p}^p. \tag{4.13}$$

In particular,

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq |a|^{2p} \|(R_H(a) - R_{H_0}(a))_-\|_{\mathcal{S}_p}^p. \tag{4.14}$$

Remark 4.1.8. Note that $\sigma(H) = \sigma_d(H) \dot\cup [0, \infty)$ where $\sigma_d(H) \subset (a, 0)$. □

Proof of Corollary 4.1.7. Let $A_0 = R_{H_0}(a)$ and $A = R_H(a)$. Then Proposition 1.1.6 implies that $\sigma(A_0) = [a^{-1}, 0]$, and Weyl's theorem shows that $\sigma(A) = \sigma_d(A) \dot\cup [a^{-1}, 0]$ where $\sigma_d(A) \subset (-\infty, a^{-1})$. Hence, by Theorem 4.1.5 we have

$$\sum_{\mu \in \sigma_d(R_H(a)), \mu < a^{-1}} (a^{-1} - \mu)^p \leq \|(R_H(a) - R_{H_0}(a))_-\|_{\mathcal{S}_p}^p$$

and another application of Proposition 1.1.6 shows the validity of (4.13). ■

The previous corollary provides the announced improvement of Theorem 3.3.1 in the selfadjoint setting. However, we will not provide a detailed comparison of these two results since it would be completely analogous to the comparison of Theorem 4.1.5 and Corollary 3.2.5 given above.

Remark 4.1.9. Given the assumptions of Corollary 4.1.7, if $H = H_0 + M$ where M is H_0-compact, then for $a < \inf \sigma(H_0 - M_-)$ we have

$$(R_H(a) - R_{H_0}(a))_- \leq R_{H_0 - M_-}(a) M_- R_{H_0}(a). \tag{4.15}$$

In particular, we see that $\|(R_H(a) - R_{H_0}(a))_-\|_{\mathcal{S}_p}^p \leq \|R_{H_0-M_-}(a)M_-R_{H_0}(a)\|_{\mathcal{S}_p}^p$. The validity of (4.15) can be seen as follows: To begin, we write

$$R_H(a) - R_{H_0}(a) = [R_H(a) - R_{H_0-M_-}(a)] - [R_{H_0}(a) - R_{H_0-M_-}(a)]. \tag{4.16}$$

Since $H \geq H_0 - M_-$ and $H_0 \geq H_0 - M_-$, the fact that $s \mapsto -\frac{1}{s}$ is operator monotone on $(0,\infty)$ (see, e.g., (Bhatia 1997), p.114) shows that the two resolvent differences in (4.16) are non-negative operators. But this implies that

$$(R_H(a) - R_{H_0}(a))_- \leq R_{H_0}(a) - R_{H_0-M_-}(a) = R_{H_0-M_-}(a)M_-R_{H_0}(a).$$

□

While we have not provided estimates in terms of semigroup differences in our consideration of non-selfadjoint operators, we would like to provide such an estimate in the selfadjoint case. To this end, let us recall that for a selfadjoint and lower-semibounded operator H in $\mathcal{H}$ the family $\{e^{-tH}\}_{t\geq 0}$ defines a **strongly continuous semigroup** on $\mathcal{H}$. That is, for every $f \in \mathcal{H}$ the mapping $\mathbb{R}_+ \ni t \mapsto e^{-tH}f \in \mathcal{H}$ is continuous and $e^{-(t+s)H} = e^{-tH}e^{-sH}$. Moreover, we have the following spectral mapping theorems:

$$\sigma(e^{-tH}) \setminus \{0\} = \{e^{-t\lambda} : \lambda \in \sigma(H)\} \quad \text{and} \quad \sigma_d(e^{-tH}) \setminus \{0\} = \{e^{-t\lambda} : \lambda \in \sigma_d(H)\}.$$

More precisely, if $\lambda \in \sigma_d(H)$ then $m_H(\lambda) = m_{e^{-tH}}(e^{-t\lambda})$, where we recall that $m_H(\lambda)$ denotes the multiplicity of λ as an eigenvalue of H.

Corollary 4.1.10. *Let H_0 and H be selfadjoint operators in $\mathcal{H}$. In addition, assume that $\sigma(H_0) = [0,\infty)$, that H is bounded from below and that $e^{-tH} - e^{-tH_0} \in \mathcal{S}_p(\mathcal{H})$ for some positive p and t. Then*

$$\sum_{\lambda \in \sigma_d(H)} (e^{-t\lambda} - 1)^p \leq \|(e^{-tH} - e^{-tH_0})_+\|_{\mathcal{S}_p}^p. \tag{4.17}$$

In particular,

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq t^{-p}\|(e^{-tH} - e^{-tH_0})_+\|_{\mathcal{S}_p}^p. \tag{4.18}$$

Remark 4.1.11. The spectral mapping theorem implies that $\sigma(e^{-tH_0}) = [0,1]$. Hence, Weyl's theorem shows that $\sigma(e^{-tH}) = [0,1] \,\dot\cup\, \sigma_d(e^{-tH})$ where $\sigma_d(e^{-tH}) \subset (1,\infty)$, and so another application of the spectral mapping theorem implies that $\sigma(H) = \sigma_d(H)\dot\cup[0,\infty)$. □

Proof of Corollary 4.1.10. Let $A_0 = e^{-tH_0}$ and $A = e^{-tH}$. Then the previous remark and Theorem 4.1.5 imply that

$$\sum_{\mu \in \sigma_d(A), \mu > 1} (\mu - 1)^p \leq \|(A - A_0)_+\|_{\mathcal{S}_p}^p.$$

Hence, applying the spectral mapping theorem and using that $e^x - 1 \geq x$ if $x > 0$, we obtain

$$t^p \sum_{\lambda \in \sigma_d(H), \lambda < 0} |\lambda|^p \leq \sum_{\lambda \in \sigma_d(H), \lambda < 0} (e^{-t\lambda} - 1)^p \leq \|(e^{-tH} - e^{-tH_0})_+\|_{\mathcal{S}_p}^p.$$

■

Let us conclude this chapter by noting that the estimates established in the previous corollary improve upon estimates derived and discussed in (Hansmann 2007), see also (Demuth & Katriel 2008). There, the validity of the inequalities

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C(p) t^{-p} \| e^{-tH} - e^{-tH_0} \|_{\mathcal{S}_p}^p, \quad p \geq 1, \tag{4.19}$$

and

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C(p,H) t^{-p} \| e^{-tH} - e^{-tH_0} \|_{\mathcal{S}_p}^p, \quad 0 < p < 1, \tag{4.20}$$

was shown using more indirect methods. For instance, inequality (4.19) (being due to Solomyak) has been derived applying operator inequalities of the type

$$\| f(A) - f(A_0) \|_{\mathcal{S}_p} \leq C(f,p) \| A - A_0 \|_{\mathcal{S}_p},$$

which are valid given suitable assumptions on the function f (see, e.g., (Birman & Solomyak 2003)). Similarly, (4.20) has been obtained applying inequalities related to Krein's spectral shift function.

As compared to (4.18), the main defect of (4.19) and (4.20) is that the constants $C(p)$ and $C(p,H)$ (apart from the cases $p = 1$ and $p = 2$) are not even close to the optimal value 1. For instance, $C(p) \to \infty$ if $p \to \infty$.

Part II.

Applications

5. Jacobi operators

In this chapter, which is based on the joint work (Hansmann & Katriel 2009), we apply the results of Section 3.2 to obtain estimates on the discrete spectrum of complex Jacobi operators.

5.1. Introduction and overview

Summary: We introduce Jacobi operators and discuss some known estimates on the discrete spectrum of compact perturbations of the free Jacobi operator.

Given three bounded complex sequences $\{a_k\}_{k\in\mathbb{Z}}, \{b_k\}_{k\in\mathbb{Z}}$ and $\{c_k\}_{k\in\mathbb{Z}}$, we define the associated **Jacobi operator** $J = J(a_k, b_k, c_k) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ as

$$(Ju)(k) = a_{k-1}u(k-1) + b_k u(k) + c_k u(k+1), \quad u \in l^2(\mathbb{Z}). \tag{5.1}$$

Then J is a bounded operator on $l^2(\mathbb{Z})$ with

$$\|J\| \leq \sup_k |a_k| + \sup_k |b_k| + \sup_k |c_k|,$$

and with respect to the standard basis $\{\delta_k\}_{k\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$, i.e., $\delta_k(j) = 0$ if $j \neq k$ and $\delta_k(k) = 1$, J can be represented by the two-sided infinite tridiagonal matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_0 & c_0 & & & \\ & & a_0 & b_1 & c_1 & & \\ & & & a_1 & b_2 & c_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

In view of this representation it is also customary to refer to J as a **Jacobi matrix**.

Example 5.1.1. The *discrete Laplace operator* on $l^2(\mathbb{Z})$ coincides with the Jacobi operator $J(1, -2, 1)$. Similarly, the Jacobi operator $J(-1, 2 + d_k, -1)$ describes a *discrete Schrödinger operator*.[1] Physically, these operators provide a discrete model of a one-dimensional quantum mechanical particle, subject to the external potential $d = \{d_k\}_{k\in\mathbb{Z}}$. □

[1] Their continuous counterparts, that is, Schrödinger operators in $L^2(\mathbb{R}^d)$, will be discussed in detail in the next chapter.

Remark 5.1.2. While we restrict ourselves to *whole line* Jacobi operators, it should at least be mentioned that, due to their intimate relation to orthogonal polynomials (see e.g. (Beckermann 2001) and references therein), it can also be of interest to study *half line* operators, i.e., Jacobi operators acting on $l^2(\mathbb{N})$ given by

$$\begin{aligned}(Ju)(1) &= b_1u(1) + c_1u(2),\\ (Ju)(k) &= a_{k-1}u(k-1) + b_ku(k) + c_ku(k+1), \quad k \geq 2.\end{aligned}$$

□

In the following we will focus on Jacobi operators which are perturbations of the **free Jacobi operator** $J_0 = J(1, 0, 1)$, i.e.,

$$(J_0u)(k) = u(k-1) + u(k+1), \quad u \in l^2(\mathbb{Z}). \tag{5.2}$$

More precisely, if $J = J(a_k, b_k, c_k)$ is defined as above, then throughout this chapter we assume that $J - J_0$ is compact.

Proposition 5.1.3. *The operator $J - J_0$ is compact if and only if*

$$\lim_{|k|\to\infty} a_k = \lim_{|k|\to\infty} c_k = 1 \quad \text{and} \quad \lim_{|k|\to\infty} b_k = 0. \tag{5.3}$$

Proof. Clearly, if (5.3) is satisfied then $J - J_0$ is a norm limit of finite rank operators and hence compact. On the other hand, if $J - J_0$ is compact then it maps weakly convergent zero-sequences into norm convergent zero-sequences. In particular,

$$\|(J - J_0)\delta_k\|_{l^2}^2 = |a_k - 1|^2 + |b_k|^2 + |c_{k-1} - 1|^2 \overset{|k|\to\infty}{\longrightarrow} 0$$

as desired. ■

Let $F : l^2(\mathbb{Z}) \to L^2(0, 2\pi)$ denote the Fourier transform, that is,

$$(Fu)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{ik\theta} u_k.$$

Then for $u \in l^2(\mathbb{Z})$ and $\theta \in [0, 2\pi)$ we have

$$(FJ_0u)(\theta) = 2\cos(\theta)(Fu)(\theta), \tag{5.4}$$

as a short computation shows. In particular, we see that J_0 is unitarily[2] equivalent to the operator of multiplication by the function $2\cos(\theta)$ on $L^2(0, 2\pi)$ and so the spectrum of J_0 coincides with the interval $[-2, 2]$. Consequently, the compactness of $J - J_0$ and Remark 1.2.9 imply that

$$\sigma(J) = [-2, 2] \,\dot{\cup}\, \sigma_d(J),$$

[2] $Z \in \mathcal{B}(\mathcal{H})$ is called unitary if $Z^* = Z^{-1}$.

so the isolated eigenvalues of J are situated in $\mathbb{C} \setminus [-2,2]$ and can accumulate on $[-2,2]$ only.

As in the previous chapters, we would like to derive estimates on $\sigma_d(J)$ given the stronger assumption that $J - J_0 \in \mathcal{S}_p$ (for simplicity, in this chapter we set $\mathcal{S}_p = \mathcal{S}_p(l^2(\mathbb{Z}))$. To this end, let us define a sequence $v = \{v_k\}_{k\in\mathbb{Z}}$ by setting

$$v_k = \max\Big(|a_{k-1}-1|, |a_k - 1|, |b_k|, |c_{k-1}-1|, |c_k - 1|\Big). \tag{5.5}$$

Clearly, the compactness of $J - J_0$ is equivalent to v_k converging to 0. Moreover, for $p \geq 1$ we will show in Lemma 5.2.6 below that $J - J_0 \in \mathcal{S}_p$ if and only if $v \in l^p(\mathbb{Z})$, and the $\mathcal{S}_p$-norm of $J - J_0$ and the l^p-norm of v are equivalent.

Remark 5.1.4. If $p \in (0,1)$ then the $\mathcal{S}_p$-norm of $J - J_0$ and the l^p-norm of v are still equivalent in the diagonal case when $a_k = c_k \equiv 1$. In general, however, all we can say is that $\|J - J_0\|_{\mathcal{S}_p} \leq 3\|v\|_{l^p}$, see Lemma 5.2.6. □

In the following, we will review some known results relating the behavior of the sequence v to the distribution of the discrete spectrum of J. To begin, let us mention the following estimate for selfadjoint Jacobi operators (that is, $b_k \in \mathbb{R}$ and $a_k = \overline{c}_k$ for all k), see Theorem 2 in (Hundertmark & Simon 2002).

Theorem 5.1.5. *Let J be selfadjoint and assume that $v \in l^p(\mathbb{Z})$ where $p \geq 1$. Then*

$$\sum_{\lambda \in \sigma_d(J),\, \lambda < -2} |\lambda + 2|^{p-\frac{1}{2}} + \sum_{\lambda \in \sigma_d(J),\, \lambda > 2} |\lambda - 2|^{p - \frac{1}{2}} \leq C(p)\|v\|_{l^p}^p.^{3} \tag{5.6}$$

Remark 5.1.6. Estimate (5.6) is usually referred to as a *Lieb-Thirring inequality* for Jacobi operators. The reason for this terminology will become clear in the next chapter, where we consider the actual Lieb-Thirring inequalities for Schrödinger operators in $L^2(\mathbb{R}^d)$. □

It should be noted that Hundertmark and Simon formulated their results for half line Jacobi operators. However, they also showed that inequality (5.6) is true for the half line operator if and only if it is true for the whole line operator. The proof of this equivalence relies on the variational characterization of the discrete spectrum.

Inequality (5.6) is optimal in the following sense: In general, it is not possible to replace the exponent $p - \frac{1}{2}$ on the left-hand side of (5.6) by any smaller exponent. For instance, this follows from a consideration of the half line operator $J(1, 1/k, 1)$ whose discrete spectrum contains the sequence

$$2\left(1 + \frac{1}{4(k+1)^2}\right)^{1/2}, \quad k \in \mathbb{N},$$

[3] Hundertmark and Simon proved this inequality for real Jacobi matrices, that is, assuming that $a_k, b_k, c_k \in \mathbb{R}$ and $a_k = c_k$. However, their proof goes through in the general selfadjoint case as well.

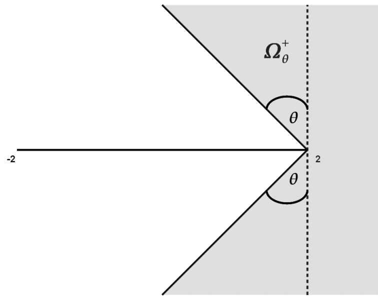

Figure 5.1.: A sketch of Ω_θ^+.

see (Kvitsinsky 1992). In addition, Example 4.3 in (Hundertmark & Simon 2002) shows that in general it is not possible to bound the sum on the left-hand side of (5.6) in terms of $\|v\|_{l^p}^p$ if p is smaller than 1.

Next, let us consider a first generalization of Theorem 5.1.5 to non-selfadjoint operators. To this end we set

$$\Omega_\theta^\pm = \{\lambda \in \mathbb{C} : 2 \mp \operatorname{Re}(\lambda) < \tan(\theta)|\operatorname{Im}\lambda|\}, \quad \theta \in [0, \pi/2),$$

see Figure 5.1.

Theorem 5.1.7. *Let $\theta \in [0, \pi/2)$. Then for $p \geq 3/2$ the following holds:*

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p-\frac{1}{2}} + \sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^-} |\lambda + 2|^{p-\frac{1}{2}} \leq C(p, \theta)\|v\|_{l^p}^p, \tag{5.7}$$

where $C(p, \theta) = C(p)(1 + 2\tan(\theta))^p$.

For a proof we refer to (Golinskii & Kupin 2007), Theorem 1.5.

Clearly, when restricted to selfadjoint operators the last theorem gives (5.6) (note, however, that in (5.7) we need $p \geq 3/2$). This is not a coincidence but due to the fact that its proof is obtained by a reduction to the case of selfadjoint operators and employing (5.6). We see that the sum in (5.7) does not involve all eigenvalues since it excludes a diamond-shaped region around the interval $[-2, 2]$, thus avoiding sequences of eigenvalues converging to some point in $(-2, 2)$. Moreover, while $\Omega_\theta^+ \cup \Omega_\theta^- \to \mathbb{C} \setminus [-2, 2]$ if $\theta \to \pi/2$, the constant $C(p, \theta)$ on the right-hand side of (5.7) diverges in this limit. Hence, rewriting inequality (5.6) in the form

$$\sum_{\lambda \in \sigma_d(J)} \operatorname{dist}(\lambda, [-2, 2])^{p-\frac{1}{2}} \leq C(p)\|v\|_{l^p}^p,$$

the last theorem indicates that this inequality might not be true for non-selfadjoint operators, due to a different behavior of sequences of eigenvalues when converging to ± 2 or $(-2, 2)$, respectively.

A further and major step in the analysis of the discrete spectrum of complex Jacobi operators was made by Borichev, Golinskii and Kupin. Transferring the analysis of the discrete spectrum of J to an analysis of the zero set of the corresponding perturbation determinant, and using Theorem 2.4.1 to study these zeros, they obtained an estimate taking into account the entire discrete spectrum of J, see Theorem 2.3 in (Borichev et al. 2009).

Theorem 5.1.8. *Let $p \geq 1$. Then for every $\tau > 0$ we have*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2,2])^{p+1+\tau}}{|\lambda^2 - 4|} \leq C(\tau, \|J - J_0\|, p) \|v\|_{l_p}^p. \tag{5.8}$$

Remark 5.1.9. Similar to Theorem 5.1.5, the previous theorem was originally established for Jacobi operators on $l^2(\mathbb{N})$. However, as we will see below, it carries over to the whole line case. Borichev et al. also derived a more refined estimate in case $p = 1$, but here their proof seems to use special properties of the half line operator and is thus not directly transferable to the whole line setting. $\square$

Comparing estimates (5.7) and (5.8) we see that both results have certain advantages and disadvantages: As already mentioned, the major advantage of (5.8) is that it provides an estimate on the entire discrete spectrum of J, while (5.7) only takes into account eigenvalues outside a diamond-shaped region around $[-2, 2]$. Moreover, while estimate (5.7) requires that $p \geq 3/2$, estimate (5.8) is valid for every $p \geq 1$, just like the corresponding estimate in the selfadjoint case.

Regarding the disadvantages of (5.8), we note that with respect to sequences converging to ± 2 from within the sets $\Omega_\theta^\pm$, estimate (5.7) is stronger than estimate (5.8). For instance, if $v \in l^p(\mathbb{Z})$ where $p \geq 3/2$ and $\theta \in [0, \pi/2)$, then (5.7) allows to conclude the finiteness of

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p - \frac{1}{2}},$$

while (5.8) only implies the finiteness of

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p+\tau}$$

where $\tau > 0$ is arbitrarily small. Finally, we note that estimate (5.8) is of a more qualitative nature than estimate (5.7) since the constant on the right-hand side of (5.8) still depends on the operator J.

In the next section, by applying the results and methods established in Section 3.2, we will see that it is possible to obtain an estimate on the discrete spectrum of J which combines the advantages of *both* estimate (5.7) and estimate (5.8).

5.2. New estimates on the discrete spectrum

Summary: We state and prove new estimates on the discrete spectrum of non-selfadjoint Jacobi operators and make a comparison with the known estimates discussed in Section 5.1.

5.2.1. Statement of results

Let us begin this section with a slight improvement of Theorem 5.1.8 (throughout this section, we use the notation of Section 5.1).

Theorem 5.2.1. *Let $\tau \in (0,1)$. If $p \geq 1-\tau$ then*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2,2])^{p+1+\tau}}{|\lambda^2-4|} \leq C(\tau,p)\|v\|_{l_p}^p. \tag{5.9}$$

Moreover, if $p \in (0, 1-\tau)$ then

$$\sum_{\lambda \in \sigma_d(J)} \left(\frac{\operatorname{dist}(\lambda, [-2,2])}{|\lambda^2-4|^{1/2}} \right)^{p+1+\tau} \leq C(\tau,p)\|v\|_{l_p}^p. \tag{5.10}$$

Proof. The above estimates are a consequence of Corollary 3.2.5 and the fact that $\|J-J_0\|_{\mathbf{S}_p}^p \leq 3^p \|v\|_{l^p}^p$, see Lemma 5.2.6 below. ∎

We note that Theorem 5.2.1 differs from Theorem 5.1.8 both in the hypothesis, which requires that $\tau \in (0,1)$, and in the conclusion, providing an estimate that depends on J only through $\|v\|_{l_p}$.

Remark 5.2.2. In view of (5.10) let us emphasize that this estimate is *not* contradictory to the fact that in the selfadjoint case inequality (5.6) cannot be true if p is smaller than 1. Indeed, if J is selfadjoint and $v \in l^p(\mathbb{Z})$ where $p \in (0, 1-\tau)$, then (5.10) only implies the finiteness of

$$\sum_{\lambda \in \sigma_d(J), \lambda < -2} |\lambda + 2|^{\frac{p+1+\tau}{2}} + \sum_{\lambda \in \sigma_d(J), \lambda > 2} |\lambda - 2|^{\frac{p+1+\tau}{2}},$$

which is fine since $\frac{1}{2}(p+1+\tau)$ is clearly larger than $p - \frac{1}{2}$. □

The following theorem is the main result of this chapter.

Theorem 5.2.3. *Let $\tau \in (0,1)$. If $v \in l^p(\mathbb{Z})$, where $p > 1$, then*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2-4|^{1/2}} \leq C(p,\tau)\|v\|_{l^p}^p. \tag{5.11}$$

Furthermore, if $v \in l^1(\mathbb{Z})$ then

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2,2])^{1+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\frac{\tau}{4}}} \leq C(\tau)\|v\|_{l^1}. \tag{5.12}$$

Let us compare the previous theorem with Theorem 5.1.7 and 5.2.1, respectively. To begin, we note that a direct calculation shows that for $\tau \in (0,1)$, $\lambda \in \mathbb{C} \setminus [-2,2]$ and $p > 1$ we have

$$\frac{\operatorname{dist}(\lambda,[-2,2])^{p+1+\tau}}{|\lambda^2-4|} \leq \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{1/2}}.$$

Moreover, if $\lambda \in \mathbb{C} \setminus [-2,2]$ and $|\lambda| \leq \|J\|$, then

$$\frac{\operatorname{dist}(\lambda,[-2,2])^{2+\tau}}{|\lambda^2-4|} \leq C(\tau,\|J\|)\frac{\operatorname{dist}(\lambda,[-2,2])^{1+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\frac{\tau}{4}}}.$$

Hence, inequalities (5.11) and (5.12) provide more information on the discrete spectrum of J than inequality (5.9), that is, Theorem 5.2.3 is stronger than Theorem 5.2.1.

To compare Theorem 5.2.3 with Theorem 5.1.7, we note that for $\theta \in [0,\pi/2)$ and $\lambda \in \Omega_\theta^\pm$, with $|\lambda| \leq \|J\|$, a direct calculation shows that

$$C(\theta,p)\frac{\operatorname{dist}(\lambda,[-2,2])^{p}}{|\lambda^2-4|^{1/2}} \leq |\lambda \mp 2|^{p-\frac{1}{2}} \leq C(\theta,p,\|J\|)\frac{\operatorname{dist}(\lambda,[-2,2])^{p}}{|\lambda^2-4|^{1/2}}.$$

Hence, ignoring the τ-corrections in the exponents of (5.11) and (5.12), we see that Theorem 5.2.3 and Theorem 5.1.7 provide exactly the same estimates on the eigenvalues of J in $\Omega_\theta^\pm$. However, we emphasize that Theorem 5.2.3 is *not* restricted to $p \in (3/2,\infty)$.

As in the more abstract estimates considered in Chapter 3, it remains an interesting open question whether the inequalities in Theorem 5.2.3 remain true in case $\tau = 0$. In particular, in view of Theorem 5.1.7 we see that a negative answer to this question can only be due to the behavior of sequences of eigenvalues converging to some point in $(-2,2)$, or due to the behavior of sequences converging to ± 2 tangentially (with respect to the interval $[-2,2]$). Moreover, it would also be interesting to know whether the exponents p and $1/2$ in the nominator and denominator of (5.11) are optimal, or whether it is possible to simultaneously replace them by $p-s$ and $\frac{1}{2}-s$ for some $s \in (0,1/2]$ (we conjecture that such a replacement is not possible).

Remark 5.2.4. We haven't touched upon the question whether Theorem 5.2.3 remains valid for half line Jacobi operators. While this would certainly be interesting to know, and the methods established in Section 3.2 can be used to study such operators as well, we will not provide the necessary computations in this thesis. A similar remark applies to higher dimensional Jacobi operators. □

Before turning to the proof of Theorem 5.2.3, which will be provided in the next section, let us conclude this section with the observation that inequalities similar to (5.11) and (5.12), but valid for a smaller range of p, can also be derived directly by means of Theorem 5.1.7. More precisely, we shall show that by a suitable integration the inequalities provided in this theorem can be used to derive the following result.

Theorem 5.2.5. *Let $\tau \in (0,1)$. If $v \in l^p(\mathbb{Z})$, where $p \geq 3/2$, then*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|v\|_{l^p}^p. \tag{5.13}$$

We remark that the proof of Theorem 5.2.5, which will be presented below, does not rely on any of the results developed in Chapter 3 and is thus completely different from the proof of Theorem 5.2.3. Note that inequality (5.13) is in fact somewhat stronger than (5.11), because of the τ in the denominator of (5.13). However, while Theorem 5.2.5 requires the condition $p \geq 3/2$, Theorem 5.2.3 is valid for $p \geq 1$, just like the corresponding inequality (5.6) in the selfadjoint case.

Proof of Theorem 5.2.5. Let $p \geq 3/2$ and $\tau \in (0,1)$. From (5.7) we know that for $\theta \in [0, \pi/2)$

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p-\frac{1}{2}} \leq C(p)(1 + 2\tan(\theta))^p \|v\|_{l^p}^p, \tag{5.14}$$

where $\Omega_\theta^+ = \{\lambda : 2 - \mathrm{Re}(\lambda) < \tan(\theta)|\operatorname{Im}\lambda|\}$. We define

$$\Psi_1 = \{\lambda : \mathrm{Re}(\lambda) > 0,\ 2 - \mathrm{Re}(\lambda) < |\operatorname{Im}(\lambda)|\} \subset \Omega_{\pi/4}^+.$$

Then a short calculation shows that for $\lambda \in \Psi_1$ we have

$$|\lambda - 2|^{p-\frac{1}{2}} \geq C(\tau) \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}}, \tag{5.15}$$

so (5.14) implies that

$$\sum_{\lambda \in \sigma_d(J) \cap \Psi_1} \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}} \leq C(p,\tau) \|v\|_{l^p}^p. \tag{5.16}$$

Let $\Psi_2 = \{\lambda : \mathrm{Re}(\lambda) > 0\} \setminus \Psi_1$ and set $x = \tan(\theta) \in [0, \infty)$. From (5.14) we obtain

$$\sum_{\lambda \in \sigma_d(J) \cap \Psi_2,\ \frac{2 - \mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|} < x} |\lambda - 2|^{p-\frac{1}{2}} \leq C(p)(1 + 2x)^p \|v\|_{l^p}^p. \tag{5.17}$$

Next, we multiply both sides of (5.17) with $x^{-p-1-\tau}$ and integrate with respect to $x \in [1, \infty)$. For the left-hand side we obtain

$$\begin{aligned}
&\int_1^\infty dx\ x^{-p-1-\tau} \sum_{\lambda \in \sigma_d(J) \cap \Psi_2,\ \frac{2 - \mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|} < x} |\lambda - 2|^{p-\frac{1}{2}} \\
=\ & \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \int_{\max(1, \frac{2 - \mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|})}^\infty dx\ x^{-p-1-\tau} \\
=\ & C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \left(\frac{|\operatorname{Im}\lambda|}{2 - \mathrm{Re}(\lambda)} \right)^{p+\tau} \\
=\ & C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \left(\frac{\mathrm{dist}(\lambda, [-2,2])}{2 - \mathrm{Re}(\lambda)} \right)^{p+\tau} \\
\geq\ & C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}}.
\end{aligned}$$

Similarly, for the right-hand side of (5.17) we obtain

$$C(p)\|v\|_{l^p}^p \int_1^\infty dx\, x^{-p-1-\tau}(1+2x)^p \leq C(p,\tau)\|v\|_{l^p}^p.$$

Hence, we have shown that

$$\sum_{\lambda\in\sigma_d(J)\cap\Psi_2} \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|v\|_{l^p}^p. \tag{5.18}$$

Noting that $\Psi_1 \cup \Psi_2 = \{\lambda : \operatorname{Re}(\lambda) > 0\}$ we can conclude from (5.16) and (5.18) that

$$\sum_{\lambda\in\sigma_d(J),\operatorname{Re}(\lambda)>0} \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|v\|_{l^p}^p.$$

Finally, starting with the estimate

$$\sum_{\lambda\in\sigma_d(J)\cap\Omega_\theta^-} |\lambda+2|^{p-\frac{1}{2}} \leq C(p)(1+2\tan(\theta))^p\|v\|_{l^p}^p,$$

which follows from (5.7), we can show in exactly the same manner as above that

$$\sum_{\lambda\in\sigma_d(J),\operatorname{Re}(\lambda)\leq 0} \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|v\|_{l^p}^p.$$

This concludes the proof of Theorem 5.2.5.

■

5.2.2. Proof of Theorem 5.2.3

Throughout this section we use the notation of Section 5.1. To begin, let the multiplication operator $M_v \in \mathcal{B}(l^2(\mathbb{Z}))$ be defined by $M_v\delta_k = v_k\delta_k$, where the sequence $v = \{v_k\}$ was defined in (5.5), i.e.,

$$v_k = \max(|a_{k-1}-1|, |a_k-1|, |b_k|, |c_{k-1}-1|, |c_k-1|),$$

and where $\{\delta_k\}$ denotes the standard basis of $l^2(\mathbb{Z})$. Furthermore, we define the operator $U \in \mathcal{B}(l^2(\mathbb{Z}))$ by setting

$$U\delta_k = u_k^-\delta_{k-1} + u_k^0\delta_k + u_k^+\delta_{k+1},$$

where (using the convention that $\frac{0}{0} = 1$)

$$u_k^- = \frac{c_{k-1}-1}{\sqrt{v_{k-1}v_k}}, \quad u_k^0 = \frac{b_k}{v_k} \quad \text{and} \quad u_k^+ = \frac{a_k-1}{\sqrt{v_{k+1}v_k}}.$$

A short calculation shows that

$$J - J_0 = M_{v^{1/2}} U M_{v^{1/2}} \tag{5.19}$$

where $v^{1/2} = \{v_k^{1/2}\}$. Moreover, the definition of $\{v_k\}$ implies that

$$|u_k^-| \leq 1, \quad |u_k^0| \leq 1 \quad \text{and} \quad |u_k^+| \leq 1,$$

showing that $\|U\| \leq 3$. The following lemma is a variation on Theorem 2.4 in (Killip & Simon 2003).

Lemma 5.2.6. *Let $p > 0$. Then*

$$\|J - J_0\|_{\mathbf{S}_p} \leq 3\|v\|_{l^p}. \tag{5.20}$$

Moreover, if $p \geq 1$ then

$$6^{-1/p}\|v\|_{l^p} \leq \|J - J_0\|_{\mathbf{S}_p}. \tag{5.21}$$

Proof. Let $p > 0$. Applying Hölder's inequality for Schatten norms (see Proposition 1.3.1) we obtain from (5.19) that

$$\begin{aligned} \|J - J_0\|_{\mathbf{S}_p} &= \|M_{v^{1/2}} U M_{v^{1/2}}\|_{\mathbf{S}_p} \leq \|M_{v^{1/2}}\|_{\mathbf{S}_{2p}} \|U M_{v^{1/2}}\|_{\mathbf{S}_{2p}} \\ &\leq 3\|M_{v^{1/2}}\|_{\mathbf{S}_{2p}}^2 = 3\|v\|_{l^p}, \end{aligned}$$

where the last equality is valid since the diagonal operator $M_{v^{1/2}}$ is selfadjoint and non-negative with eigenvalues $v_k^{1/2}$. To show the remaining inequality, we use that for $p \geq 1$

$$\sum_{k \in \mathbb{Z}} [|a_k - 1|^p + |b_k|^p + |c_k - 1|^p] \leq 3\|J - J_0\|_{\mathbf{S}^p}^p,^4$$

see (Killip & Simon 2003), Lemma 2.3 (iii) and its proof. Since

$$\begin{aligned} \|v\|_{l^p}^p &= \sum_{k \in \mathbb{N}} \max\Big(|a_{k-1} - 1|^p, |a_k - 1|^p, |b_k|^p, |c_{k-1} - 1|^p, |c_k - 1|^p\Big) \\ &\leq 2\sum_{k \in \mathbb{N}} [|a_k - 1|^p + |b_k|^p + |c_k - 1|^p], \end{aligned}$$

we obtain (5.21). ∎

In the following we intend to prove Theorem 5.2.3 by an application of Theorem 3.2.3. Since we have seen above that $J - J_0 = M_{v^{1/2}} U M_{v^{1/2}}$, we will apply Thm. 3.2.3 choosing (with the notation of that theorem) $M_1 = M_{v^{1/2}}$ and $M_2 = U M_{v^{1/2}}$, and so we need an appropriate bound on the Schatten norm of $U M_{v^{1/2}} R_{J_0}(\lambda) M_{v^{1/2}}$.

Lemma 5.2.7. *Let $v \in l^p(\mathbb{Z})$ where $p \geq 1$. Then the following holds: If $p > 1$ then*

$$\|U M_{v^{1/2}} R_{J_0}(\lambda) M_{v^{1/2}}\|_{\mathbf{S}_p}^p \leq \frac{C(p)\|v\|_{l^p}^p}{\operatorname{dist}(\lambda, [-2,2])^{p-1} |\lambda^2 - 4|^{1/2}}. \tag{5.22}$$

Furthermore, if $v \in l^1(\mathbb{Z})$ then for every $\varepsilon \in (0,1)$ we have

$$\|U M_{v^{1/2}} R_{J_0}(\lambda) M_{v^{1/2}}\|_{\mathbf{S}_1} \leq \frac{C(\varepsilon)\|v\|_{l^1}}{\operatorname{dist}(\lambda, [-2,2])^{\varepsilon} |\lambda^2 - 4|^{(1-\varepsilon)/2}}. \tag{5.23}$$

[4] We don't know whether a similar inequality holds for $p \in (0,1)$ as well.

The proof of Lemma 5.2.7 will be given below. First, let us continue with the proof of Theorem 5.2.3. To this end, let us assume that $v \in l^p(\mathbb{Z})$ and let us fix $\tau \in (0,1)$. Considering the case $p > 1$ first, we obtain from (5.22) and Theorem 3.2.3, with $\alpha = p-1$, $\beta = -1/2$ and $K = C(p)\|v\|_{l^p}^p$, i.e., $\eta_1 = p+\tau$ and $\eta_2 = p-1+\tau$,

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{1/2}} \leq C(p,\tau)\|v\|_{l^p}^p.$$

Similarly, if $p = 1$ then we obtain from (5.23) and Theorem 3.2.3 that for $\varepsilon \in (0,1)$ and $\tilde{\tau} \in (0,1)$

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda,[-2,2])^{1+\varepsilon+\tilde{\tau}}}{|\lambda^2-4|^{(1+\varepsilon)/2}} \leq C(\tilde{\tau},\varepsilon)\|v\|_{l^1}.$$

Choosing $\varepsilon = \tilde{\tau} = \tau/2$ concludes the proof of Theorem 5.2.3.

It remains to prove Lemma 5.2.7. To begin, we recall that

$$(FJ_0f)(\theta) = 2\cos(\theta)(Ff)(\theta), \quad f \in l^2(\mathbb{Z}), \quad \theta \in [0,2\pi),$$

where F denotes the Fourier transform. Consequently, for $\lambda \in \mathbb{C} \setminus [-2,2]$ we have

$$R_{J_0}(\lambda) = F^{-1}M_{g_\lambda}F,$$

where $M_{g_\lambda} \in \mathcal{B}(L^2(0,2\pi))$ is the operator of multiplication by the bounded function

$$g_\lambda(\theta) = (\lambda - 2\cos(\theta))^{-1}, \quad \theta \in [0,2\pi). \tag{5.24}$$

Since $g_\lambda = |g_\lambda|^{1/2} \cdot \frac{g_\lambda}{|g_\lambda|} \cdot |g_\lambda|^{1/2}$, we can define the unitary operator $T = F^{-1}M_{g_\lambda/|g_\lambda|}F$ to obtain the identity

$$\|UM_{v^{1/2}}R_{J_0}(\lambda)M_{v^{1/2}}\|_{\mathcal{S}_p}^p = \|UM_{v^{1/2}}F^{-1}M_{|g_\lambda|^{1/2}}FTF^{-1}M_{|g_\lambda|^{1/2}}FM_{v^{1/2}}\|_{\mathcal{S}_p}^p.$$

Using Hölder's inequality for Schatten norms (see Proposition 1.3.1), and recalling that $\|U\| \leq 3$, we thus obtain

$$\begin{aligned}
\|UM_{v^{1/2}}R_{J_0}(\lambda)M_{v^{1/2}}\|_{\mathcal{S}_p}^p &\leq 3^p\|M_{v^{1/2}}F^{-1}M_{|g_\lambda|^{1/2}}F\|_{\mathcal{S}_{2p}}^p\|F^{-1}M_{|g_\lambda|^{1/2}}FM_{v^{1/2}}\|_{\mathcal{S}_{2p}}^p \\
&= 3^p\|M_{v^{1/2}}F^{-1}M_{|g_\lambda|^{1/2}}F\|_{\mathcal{S}_{2p}}^{2p}.
\end{aligned} \tag{5.25}$$

For the last identity we used Proposition 1.3.5 and the fact that the operators $M_{v^{1/2}}$ and $F^{-1}M_{|g_\lambda|^{1/2}}F$ are bounded and selfadjoint.

To derive an estimate on the Schatten norm on the right-hand side of (5.25) we will use the following lemma. Here, as above, $M_u \in \mathcal{B}(l^2(\mathbb{Z}))$ and $M_h \in \mathcal{B}(L^2(0,2\pi))$ denote the operators of multiplication by a sequence $u = \{u_m\} \in l^\infty(\mathbb{Z})$ and a function $h \in L^\infty(0,2\pi)$, respectively.

Lemma 5.2.8. *Let $q \geq 2$ and suppose that $u = \{u_m\} \in l^q(\mathbb{Z})$ and $h \in L^\infty(0,2\pi)$. Then*

$$\|M_uF^{-1}M_hF\|_{\mathcal{S}_q} \leq (2\pi)^{-1/q}\|u\|_{l^q}\|h\|_{L^q}. \tag{5.26}$$

For operators on $L^2(\mathbb{R}^d)$ this is a well-known result, see Theorem 4.1 in (Simon 2005). Since the proofs in the discrete and continuous settings are completely analogous, we only provide a sketch.

Sketch of proof of Lemma 5.2.8. We note that $M_u F^{-1} M_h F$ is an integral operator on $l^2(\mathbb{Z})$ with kernel $(2\pi)^{-\frac{1}{2}} u_m (F^{-1}h)_{m-n}$ where $m,n \in \mathbb{Z}$. Hence, the $\mathcal{S}_2$-norm of that operator can be calculated as follows:

$$\begin{aligned} \|M_u F^{-1} M_h F\|_{\mathcal{S}_2}^2 &= (2\pi)^{-1} \sum_{m,n} |u_m (F^{-1}h)_{m-n}|^2 \\ &= (2\pi)^{-1} \|u\|_{l^2}^2 \|F^{-1}h\|_{l^2}^2 = (2\pi)^{-1} \|u\|_{l^2}^2 \|h\|_{L^2}^2. \end{aligned}$$

Moreover, for the operator norm we have

$$\|M_u F^{-1} M_h F\| \leq \|M_u\| \|F^{-1} M_h F\| = \|M_u\| \|M_h\| = \|u\|_{l^\infty} \|h\|_{L^\infty}.$$

The general result now follows by complex interpolation. For details, see the proof of Theorem 4.1 in (Simon 2005). ■

Since $p \geq 1$ the previous lemma and (5.25) imply that

$$\|U M_{v^{1/2}} R_{J_0}(\lambda) M_{v^{1/2}}\|_{\mathcal{S}_p}^p \leq C(p) \|M_{v^{1/2}} F^{-1} M_{|g_\lambda|^{1/2}} F\|_{\mathcal{S}_{2p}}^{2p} \leq C(p) \|v\|_{l^p}^p \|g_\lambda\|_{L^p}^p. \tag{5.27}$$

But now the proof of Lemma 5.2.7 is completed by an application of the following result.

Lemma 5.2.9. *Let $\lambda \in \mathbb{C} \setminus [-2,2]$ and let $g_\lambda : [0, 2\pi) \to \mathbb{C}$ be defined by (5.24). Then the following holds: If $p > 1$ then*

$$\|g_\lambda\|_{L^p}^p \leq \frac{C(p)}{\operatorname{dist}(\lambda, [-2,2])^{p-1} |\lambda^2 - 4|^{1/2}}. \tag{5.28}$$

Furthermore, for every $\varepsilon \in (0,1)$ we have

$$\|g_\lambda\|_{L^1} \leq \frac{C(\varepsilon)}{\operatorname{dist}(\lambda, [-2,2])^{\varepsilon} |\lambda^2 - 4|^{(1-\varepsilon)/2}}. \tag{5.29}$$

Proof of Lemma 5.2.9. Let us first show that (5.29) is an immediate consequence of (5.28): For $r > 1$ Hölder's inequality and (5.28) imply (remember that $L^2 = L^2(0, 2\pi)$)

$$\begin{aligned} \|g_\lambda\|_{L^1} &= \|g_\lambda \cdot 1\|_{L^1} \leq \|g_\lambda\|_{L^r} \|1\|_{L^{r/(r-1)}} \\ &\leq \frac{C(r)}{\operatorname{dist}(\lambda, [-2,2])^{1-\frac{1}{r}} |\lambda^2 - 4|^{\frac{1}{2r}}}. \end{aligned}$$

Choosing $r = \frac{1}{1-\varepsilon}$, where $0 < \varepsilon < 1$, implies the validity of (5.29). It remains to show (5.28).

Let $\lambda = w + w^{-1}$ where $w \in \mathbb{D}$, and let $m(.)$ denote normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$. Then

$$\begin{aligned} \|g_\lambda\|_{L^p}^p &= \int_0^{2\pi} \frac{d\theta}{|\lambda - 2\cos(\theta)|^p} \\ &= \int_{\mathbb{T}} \frac{m(d\xi)}{|\xi + \overline{\xi} - w - w^{-1}|^p} = |w|^p \int_{\mathbb{T}} \frac{m(d\xi)}{|\xi - w|^p |\xi - \overline{w}|^p}. \end{aligned} \tag{5.30}$$

Let $\mathbb{T}^+ = \{\xi \in \mathbb{T} : \mathrm{Im}(\xi) > 0\}$ and $\mathbb{T}^- = \mathbb{T} \backslash \mathbb{T}^+$. In the following, let us suppose that $\mathrm{Im}(w) \geq 0$ (the other case can be handled similarly). Then for $\xi \in \mathbb{T}^+$ we have

$$|\xi - \overline{w}| \geq \min(|1-\overline{w}|, |1+\overline{w}|) \geq \frac{1}{2}|1-w^2|.$$

Similarly, for $\xi \in \mathbb{T}^-$ we obtain

$$|\xi - w| \geq \min(|1-w|, |1+w|) \geq \frac{1}{2}|1-w^2|.$$

From (5.30) we can thus deduce that

$$\|g_\lambda\|_{L^p}^p \quad \leq \quad \frac{2^p|w|^p}{|1-w^2|^p}\left(\int_{\mathbb{T}^+} \frac{m(d\xi)}{|\xi - w|^p} + \int_{\mathbb{T}^-} \frac{m(d\xi)}{|\xi - \overline{w}|^p}\right). \tag{5.31}$$

For $\mu \in \mathbb{D}$ we have the estimate (recall that $p > 1$)

$$\int_{\mathbb{T}} \frac{m(d\xi)}{|\xi - \mu|^p} = O((1-|\mu|)^{1-p}), \quad \text{if } |\mu| \to 1,$$

see Lemma 2.2.7. Applying this estimate to (5.31) we obtain

$$\|g_\lambda\|_{L^p}^p \leq \frac{C(p)|w|^p}{|1-w^2|^p(1-|w|)^{p-1}}.$$

Since

$$\frac{|w^2-1|(1-|w|)}{|w|} \geq \frac{2}{1+\sqrt{2}}\,\mathrm{dist}(\lambda, [-2,2])$$

by Lemma 3.2.1, and

$$|\lambda^2 - 4|^{1/2} = \frac{|w^2-1|}{|w|},$$

we see that

$$\|g_\lambda\|_{L^p}^p \leq \frac{C(p)}{\mathrm{dist}(\lambda, [-2,2])^{p-1}|\lambda^2-4|^{1/2}}.$$

This concludes the proof of Lemma 5.2.9. ■

Remark 5.2.10. In this chapter we have finally seen why it was advantageous to formulate Theorem 3.2.3 in terms of estimates on $M_2 R_{A_0}(\lambda) M_1$ instead of estimates on $M R_{A_0}(\lambda)$ (where $A = A_0 + M = A_0 + M_1 M_2$). Without this decomposition the estimates in Theorem 5.2.3 could have been proved for $p \geq 2$ only, due to the restriction to such p's in Lemma 5.2.8. We will face this problem again, without being able to solve it, in our considerations of Schrödinger operators in the next chapter. □

6. Schrödinger operators

In this chapter, which is based on and extends the joint work (Demuth et al. 2009), we study the discrete spectrum of Schrödinger operators with complex potentials.

Remark. Throughout this chapter we assume some familiarity with the theory of closed sectorial forms and their associated operators. A short review of this topic is provided in Appendix B. □

6.1. Definition of the operators

Summary: We introduce Schrödinger operators $-\Delta + V$, with a complex potential V, and establish conditions on V guaranteeing that their essential spectrum coincides with the interval $[0, \infty)$.

In this chapter, $H_0 = -\Delta$ denotes the selfadjoint realization of the **Laplace operator** in $L^2(\mathbb{R}^d), d \geq 1$, that is,

$$\begin{aligned} \mathrm{Dom}(H_0) &= W^{2,2}(\mathbb{R}^d), \\ H_0 f &= F^{-1} M_{|k|^2} F f, \quad f \in \mathrm{Dom}(H_0). \end{aligned} \tag{6.1}$$

Here $M_{|k|^2}$ is the operator of multiplication by the function $\mathbb{R}^d \ni k \mapsto |k|^2 \in \mathbb{R}$ and $W^{2,2}(\mathbb{R}^d)$ denotes the **Sobolev space** of order 2, i.e.,

$$W^{s,2}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : M_{|k|^s} F f \in L^2(\mathbb{R}^d) \right\}, \quad s > 0.$$

Remark 6.1.1. We should note that in this chapter $F \in \mathcal{B}(L^2(\mathbb{R}^d))$ denotes the *continuous* Fourier transform, that is, the unique unitary extension of $F_0 : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$,

$$(F_0 f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ixy} f(y) dy,$$

defined on the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions. While the same symbol F has been used in the previous chapter to denote the *discrete* Fourier transform on $l^2(\mathbb{Z})$, we think there is no danger of confusion. □

As a direct consequence of its definition, we see that H_0 is a non-negative operator whose spectrum coincides with the interval $[0, \infty)$. Let us remark that an alternative characterization of H_0 can be established via quadratic form techniques, that is, H_0

is the unique operator associated with the densely defined, closed, sectorial (or more precisely, non-negative) form

$$\mathcal{E}_0(u,v) = \int_{\mathbb{R}^d} |k|^2 (Fu)(k)\overline{(Fv)(k)}\,dk, \quad \mathrm{Dom}(\mathcal{E}_0) = W^{1,2}(\mathbb{R}^d). \tag{6.2}$$

In the following, we will be interested in Schrödinger operators formally given as $H_0 + V$, where the **potential** V is a complex-valued function on $\mathbb{R}^d$ satisfying

$$V \in \begin{cases} L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), & \text{if } d \geq 3, \\ L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2), & \text{if } d = 2, \quad (\text{with } \varepsilon > 0 \text{ arbitrary}) \\ L^1(\mathbb{R}) + L^\infty(\mathbb{R}), & \text{if } d = 1. \end{cases} \tag{6.3}$$

Let us begin with a precise definition of these operators. To this end, for V satisfying (6.3) let us define a form $\mathcal{E}_V$ by setting

$$\begin{aligned} \mathcal{E}_V(u,v) &= \langle Vu, v\rangle = \textstyle\int_{\mathbb{R}^d} V(x)u(x)\overline{v(x)}dx, \\ \mathrm{Dom}(\mathcal{E}_V) &= \left\{u \in L^2(\mathbb{R}^d) : Vu^2 \in L^1(\mathbb{R}^d)\right\}. \end{aligned} \tag{6.4}$$

Lemma 6.1.2. *Let $\mathcal{E}_0$ and $\mathcal{E}_V$ be defined as above. Then $\mathcal{E}_V$ is $\mathcal{E}_0$-bounded with $\mathcal{E}_0$-bound zero.*

Remark 6.1.3. While the material in this section is standard, for the convenience of the reader we will provide a proof, or at least a sketch of proof, for every result to be discussed. □

Sketch of proof of Lemma 6.1.2. We restrict ourselves to the case $d \geq 3$. The case $d \leq 2$ can be treated in an analogous fashion.

Let us begin by recalling the following Sobolev inequality (to be found in, e.g., (Lieb & Loss 2001), p.202): If $u \in W^{1,2}(\mathbb{R}^d)$ then $u \in L^{2d/(d-2)}(\mathbb{R}^d)$ and

$$\mathcal{E}_0[u] := \mathcal{E}_0(u,u) \geq C_d \|u\|^2_{L^{2d/(d-2)}}, \tag{6.5}$$

where $C_d > 0$ is a constant independent of u.

Next, let $\varepsilon > 0$ be arbitrary. Since $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, it can be decomposed as $V = V_1 + V_2$, where $V_2 \in L^\infty(\mathbb{R}^d)$ and $V_1 \in L^{d/2}(\mathbb{R}^d)$ with $\|V_1\|_{L^{d/2}} \leq \varepsilon C_d$ (where C_d is taken from (6.5)). In particular, we see that for every $u \in W^{1,2}(\mathbb{R}^d)$

$$|\mathcal{E}_V[u]| \leq |\langle V_1 u, u\rangle| + |\langle V_2 u, u\rangle| \leq |\langle V_1 u, u\rangle| + \|V_2\|_{L^\infty}\|u\|^2_{L^2}. \tag{6.6}$$

Moreover, Hölder's inequality and (6.5) show that

$$\begin{aligned} |\langle V_1 u, u\rangle| &\leq \|V_1\|_{L^{d/2}} \|u^2\|_{L^{d/(d-2)}} = \|V_1\|_{L^{d/2}}\|u\|^2_{L^{2d/(d-2)}} \\ &\leq \|V_1\|_{L^{d/2}} C_d^{-1}\mathcal{E}_0[u] \leq \varepsilon\mathcal{E}_0[u]. \end{aligned} \tag{6.7}$$

Hence, (6.6) and (6.7) imply that for every $u \in W^{1,2}(\mathbb{R}^d) = \mathrm{Dom}(\mathcal{E}_0)$

$$|\mathcal{E}_V[u]| \leq \|V_2\|_{L^\infty}\|u\|^2_{L^2} + \varepsilon\mathcal{E}_0[u],$$

showing that $\mathcal{E}_V$ is form bounded with respect to $\mathcal{E}_0$. Since $\varepsilon > 0$ was arbitrary, the $\mathcal{E}_0$-bound of $\mathcal{E}_V$ is zero. ∎

If the potential V satisfies (6.3) then the previous lemma and Theorem B.4 imply that $\mathrm{Dom}(\mathcal{E}_0) \subset \mathrm{Dom}(\mathcal{E}_V)$ and that the form

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_V, \quad \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\mathcal{E}_0), \tag{6.8}$$

is densely defined, closed and sectorial, so by means of Theorem B.5 we can uniquely associate an m-sectorial operator H to this form such that $\mathrm{Dom}(H) \subset \mathrm{Dom}(\mathcal{E})$ and

$$\mathcal{E}(u, v) = \langle Hu, v\rangle$$

for every $u \in \mathrm{Dom}(H)$ and $v \in \mathrm{Dom}(\mathcal{E})$.[1] This operator H is the **Schrödinger operator** we intended to define. In the following, to indicate its origin as an operator corresponding to the form sum of $\mathcal{E}_0$ and $\mathcal{E}_V$, we will use the notation

$$H = H_0 \dot{+} M_V. \tag{6.9}$$

We note that the form sum $H_0 \dot{+} M_V$ extends the operator sum $H_0 + M_V$, where M_V is the operator of multiplication by V defined on $\mathrm{Dom}(M_V) = \{f \in L^2(\mathbb{R}^d) : M_V f \in L^2(\mathbb{R}^d)\}$. That is, $\mathrm{Dom}(H_0 + M_V) \subset \mathrm{Dom}(H_0 \dot{+} M_V)$ and $(H_0 \dot{+} M_V)|_{\mathrm{Dom}(H_0+M_V)} = H_0 + M_V$. For later purposes, let us state a simple criterion which shows when these operators coincide.

Lemma 6.1.4. *Let $H = H_0 \dot{+} M_V$ and $\tilde{H} = H_0 + M_V$ be defined as above. If $\rho(H) \cap \rho(\tilde{H}) \neq \emptyset$, then $H = \tilde{H}$.*

Proof. If $a \in \mathbb{C}$ and both $a - H$ and $a - \tilde{H}$ are injective, then $(a - H)^{-1}$ (defined on $\mathrm{Ran}(a - H)$) is an extension of $(a - \tilde{H})^{-1}$ since H is an extension of $\tilde{H}$. Hence, if $a \in \rho(H) \cap \rho(\tilde{H})$ then $R_H(a) = R_{\tilde{H}}(a)$. But this implies that $\mathrm{Dom}(H) = \mathrm{Dom}(\tilde{H})$ and so $H = \tilde{H}$. ■

Remark 6.1.5. Let $\mathrm{Num}(H) = \{\langle Hf, f\rangle : f \in \mathrm{Dom}(H), \|f\|_{L^2} = 1\}$ and $\mathrm{Num}(\mathcal{E}) = \{\mathcal{E}(u, u) : u \in \mathrm{Dom}(\mathcal{E}), \|u\|_{L^2} = 1\}$ denote the **numerical range** of H and $\mathcal{E}$, respectively. Then the sectoriality of $\mathcal{E}$ implies the existence of $\omega_0 \geq 0$ and $\theta \in [0, \pi/2)$ such that

$$\mathrm{Num}(H) \subset \mathrm{Num}(\mathcal{E}) \subset \{\lambda : |\arg(\lambda + \omega_0)| \leq \theta\}.$$

In particular, we have $\mathrm{Num}(H) \subset \mathbb{H}^+_{-\omega_0} = \{\lambda : \mathrm{Re}(\lambda) \geq -\omega_0\}$. Furthermore, we note that $\sigma(H) \subset \overline{\mathrm{Num}(H)}$ and that for $\lambda \notin \overline{\mathrm{Num}(H)}$ we have

$$\|R_H(\lambda)\| \leq \mathrm{dist}(\lambda, \overline{\mathrm{Num}(H)})^{-1}.$$

While we will not discuss the precise dependence of ω_0 and θ on V, which, for instance, can be obtained from an analysis of the proof of Lemma 6.1.2, for later purposes let us at least mention the simple fact that we can choose $\omega_0 = 0$ if $\mathrm{Re}(V) \geq 0$. □

[1] More precisely, if $u \in \mathrm{Dom}(\mathcal{E})$, $w \in L^2(\mathbb{R}^d)$ and $\mathcal{E}(u, v) = \langle w, v\rangle$ for every $v \in \mathrm{Dom}(\mathcal{E})$, then $u \in \mathrm{Dom}(H)$ and $Hu = w$.

In order to apply the results of Section 3.3 to study the discrete spectrum of H, which we will do in the sections to follow, we need to assure that $R_H(a) - R_{H_0}(a)$ is compact for some $a \in \rho(H) \cap \rho(H_0)$. In particular, by Proposition 1.2.6 this would imply that

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty).$$

However, considering the special case of a constant non-vanishing potential V, we see that assumption (6.3) is too general to allow for this conclusion. For this reason, in the following we will restrict ourselves to potentials $V \in L^p(\mathbb{R}^d)$ with

$$p \in \begin{cases} [d/2, \infty), & \text{if } d \geq 3, \\ (1, \infty), & \text{if } d = 2, \\ [1, \infty), & \text{if } d = 1. \end{cases} \tag{6.10}$$

Note that every $V \in L^p(\mathbb{R}^d)$ with p satisfying (6.10) also satisfies assumption (6.3), i.e., the operator $H = H_0 \dot{+} M_V$ is still well defined.[2] To show that the resolvent difference corresponding to H and H_0 is indeed compact for this class of potentials, we will need the following lemma.

Lemma 6.1.6. *Let $H = H_0 \dot{+} M_V$ where $V \in L^p(\mathbb{R}^d)$ with p satisfying (6.10), and let $V_n \in C_0^\infty(\mathbb{R}^d)$ with $\|V - V_n\|_{L^p} \overset{n\to\infty}{\to} 0$.[3] If $H_n = H_0 \dot{+} M_{V_n}$ then every $\lambda \in \rho(H)$ belongs to $\rho(H_n)$ for sufficiently large n and*

$$\|R_H(\lambda) - R_{H_n}(\lambda)\| \overset{n\to\infty}{\to} 0. \tag{6.11}$$

Remark 6.1.7. We note that for $W \in C_0^\infty(\mathbb{R}^d)$ the form sum $H_0 \dot{+} M_W$ coincides with the operator sum $H_0 + M_W$. For instance, this follows from Remark 6.1.5 and Lemma 6.1.4 noting that $M_W \in \mathcal{B}(L^2(\mathbb{R}^d))$ and $\sigma(H_0 + M_W) \subset \{\lambda : \operatorname{dist}(\lambda, [0, \infty)) \leq \|M_W\|\}$. □

Sketch of proof of Lemma 6.1.6. Let $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_V$ and $\mathcal{E}_n = \mathcal{E}_0 + \mathcal{E}_{V_n}$ denote the closed sectorial forms corresponding to H and H_n, respectively. Note that $\operatorname{Dom}(\mathcal{E}) = \operatorname{Dom}(\mathcal{E}_n) = W^{1,2}(\mathbb{R}^d)$. As in the proof of Lemma 6.1.2 we restrict ourselves to the case $d \geq 3$ and $p \geq d/2$.

Let us set $W_n = V - V_n$ and $c_n = \|W_n\|_{L^p}$, so $c_n \overset{n\to\infty}{\to} 0$ by assumption. It is no loss to assume that $c_n > 0$ for all n. In addition, we set

$$W_{n,1} = W_n \cdot \chi_{\{|W_n| > c_n\}} \quad \text{and} \quad W_{n,2} = W_n - W_{n,1}.$$

Clearly, $W_{n,2} \in L^\infty(\mathbb{R}^d)$ with $\|W_{n,2}\|_{L^\infty} \leq c_n$. Moreover, since $p \geq d/2$ we have

$$\begin{aligned} \|W_{n,1}\|_{L^{d/2}}^{d/2} &= \int_{|W_n| > c_n} |W_n(x)|^{d/2} dx \\ &= c_n^{d/2} \int_{c_n^{-1}|W_n| > 1} |c_n^{-1} W_n(x)|^{d/2} dx \\ &\leq c_n^{d/2} \|c_n^{-1} W_n\|_{L^p}^p = c_n^{\frac{d}{2} - p} \|W_n\|_{L^p}^p = c_n^{d/2}, \end{aligned}$$

[2] This is a consequence of the decomposition $V = V\chi_{\{|V|>1\}} + V\chi_{\{|V|\leq 1\}}$, where χ_K denotes the characteristic function of a set $K \subset \mathbb{R}^d$.

[3] Here $C_0^\infty(\mathbb{R}^d)$ denotes the class of all smooth functions $f : \mathbb{R}^d \to \mathbb{C}$ having compact support.

showing that $W_{n,1} \in L^{d/2}(\mathbb{R}^d)$ and $\|W_{n,1}\|_{L^{d/2}} \le c_n$. In the following, let $u \in W^{1,2}(\mathbb{R}^d)$. Then we obtain

$$\begin{aligned} |(\mathcal{E} - \mathcal{E}_n)[u]| &= |(\mathcal{E}_V - \mathcal{E}_{V_n})[u]| = |\langle W_n u, u\rangle| \\ &\le |\langle W_{n,1} u, u\rangle| + |\langle W_{n,2} u, u\rangle| \\ &\le |\langle W_{n,1} u, u\rangle| + \|W_{n,2}\|_{L^\infty} \|u\|_{L^2}^2 \\ &\le |\langle W_{n,1} u, u\rangle| + c_n \|u\|_{L^2}^2. \end{aligned}$$

As in the proof of Lemma 6.1.2 (see (6.7)), by means of Hölder's and Sobolev's inequalities we can estimate

$$|\langle W_{n,1} u, u\rangle| \le C_d^{-1} \|W_{n,1}\|_{L^{d/2}} \mathcal{E}_0[u] \le C_d^{-1} c_n \mathcal{E}_0[u],$$

which together with the previous estimate shows that

$$|(\mathcal{E} - \mathcal{E}_n)[u]| \le c_n \left(\|u\|_{L^2}^2 + C_d^{-1} \mathcal{E}_0[u] \right). \tag{6.12}$$

From Lemma 6.1.2 we know that $\mathcal{E}_V$ is $\mathcal{E}_0$-bounded with form bound 0. In particular, the same is true for $\operatorname{Re}(\mathcal{E}_V) = \mathcal{E}_{\operatorname{Re}(V)}$ and there exists a positive constant $C^1(V,d)$ such that

$$|\operatorname{Re}(\mathcal{E}_V)[u]| \le C^1(V,d) \|u\|_{L^2}^2 + \frac{1}{2} \mathcal{E}_0[u].$$

The last inequality implies that

$$\mathcal{E}_0[u] \le 2C^1(V,d) \|u\|_{L^2}^2 + 2(\mathcal{E}_0[u] + \operatorname{Re}(\mathcal{E}_V)[u]).$$

Noting that $\mathcal{E}_0[u] + \operatorname{Re}(\mathcal{E}_V)[u] = \operatorname{Re}(\mathcal{E})[u]$, we thus obtain from (6.12) that

$$|(\mathcal{E} - \mathcal{E}_n)[u]| \le r_n \|u\|_{L^2}^2 + s_n \operatorname{Re}(\mathcal{E})[u], \tag{6.13}$$

where $r_n = c_n \left(1 + 2C_d^{-1} C^1(V,d)\right)$ and $s_n = 2C_d^{-1} c_n$. Since $r_n, s_n \to 0$ for $n \to \infty$, an application of Theorem B.6 concludes the proof. ■

With the next proposition we obtain the desired compactness of the resolvent difference corresponding to H and H_0, given assumption (6.10).

Proposition 6.1.8. *Let $H = H_0 \dotplus M_V$, where $V \in L^p(\mathbb{R}^d)$ with p satisfying (6.10). Then for $a < 0$ with $|a|$ sufficiently large we have $a \in \rho(H) \cap \rho(H_0)$ and*

$$R_H(a) - R_{H_0}(a) \in \mathcal{S}_\infty.^4 \tag{6.14}$$

Proof. Since H is m-sectorial and H_0 is selfadjoint and non-negative, we have $a \in \rho(H_0) \cap \rho(H)$ if $a < 0$ and $|a|$ is sufficiently large. Let $V_n \in C_0^\infty(\mathbb{R}^d)$ be chosen as in Lemma 6.1.6 and let $H_n = H_0 + M_{V_n}$. By Lemma 6.1.6 we know that $a \in \rho(H_n)$ for n sufficiently large and

$$\|[R_H(a) - R_{H_0}(a)] - [R_{H_n}(a) - R_{H_0}(a)]\| = \|R_H(a) - R_{H_n}(a)\| \overset{n\to\infty}{\to} 0. \tag{6.15}$$

We will see in Lemma 6.3.10 below that for $n \in \mathbb{N}$ the operator $M_{V_n} R_{H_0}(a)$ is compact, so the second resolvent identity implies that $R_{H_n}(a) - R_{H_0}(a)$ is compact as well and the compactness of $R_H(a) - R_{H_0}(a)$ is a consequence of (6.15). ■

[4] In this chapter, for $p \in (0, \infty]$ we set $\mathcal{S}_p = \mathcal{S}_p(L^2(\mathbb{R}^d))$.

In particular, the last proposition and Proposition 1.2.8 imply that for $V \in L^p(\mathbb{R}^d)$, with p satisfying (6.10), we have

$$\sigma(H) = [0,\infty) \,\dot{\cup}\, \sigma_d(H),$$

so the isolated eigenvalues of H are situated in $\mathbb{C} \setminus [0,\infty)$ and can accumulate on $[0,\infty)$ only. In the next section, we will continue with a short overview of some known results relating the distribution of these eigenvalues to suitable L^p-bounds on the potential V.

6.2. The discrete spectrum - a short overview

Summary: We present some known estimates on the discrete spectrum of non-selfadjoint Schrödinger operators, concentrating on generalizations of the Lieb-Thirring inequalities.

Let us begin with a look at the selfadjoint case, where the potential V is a real-valued function. In this case, the operator $H = H_0 \dot{+} M_V$ describes the non-relativistic motion of a quantum mechanical particle in an external potential V, with the negative eigenvalues of H corresponding to the possible energy levels of the system. The analysis of the discrete spectrum of such operators, under various assumptions on the potential V, is one of the main subjects of mathematical physics and the amount of results on this topic is vast, see, e.g., (Reed & Simon 1978) and references therein.

In the following, we will concentrate on one particular result, that is, the celebrated **Lieb-Thirring inequalities**. They state the following: If $V = \overline{V} \in L^p(\mathbb{R}^d)$ with p satisfying (6.10) then

$$\sum_{\lambda \in \sigma_d(H), \lambda < 0} |\lambda|^{p-\frac{d}{2}} \leq C(p,d) \|V_-\|_{L^p}^p, \text{[5]} \tag{6.16}$$

where, as we recall, $V_- = -\min(V,0)$ denotes the negative part of the potential V. The inequalities (6.16) were a major tool in Lieb and Thirring's proof of the stability of matter (see (Lieb & Thirring 1975)) and the precise evaluation of the constants $C(p,d)$ remains an active field of current research. We refer to (Hundertmark 2007) for an overview on this topic.

Remark 6.2.1. The Lieb-Thirring inequalities are optimal with respect to the range of p, i.e., (6.16) can not hold for $p = 1$ and $p \in [1/2, 1)$ if $d = 2$ and $d = 1$, respectively. □

Remark 6.2.2. Since the positive part of V does not appear on the right-hand side of (6.16), we see that our assumption that $V_+ \in L^p(\mathbb{R}^d)$, with p satisfying (6.10), can be relaxed considerably. Indeed, inequality (6.16) remains valid whenever the right-hand side is finite and the form sum $H = H_0 \dot{+} M_V$ is well defined (however, we note that in order that $\sigma_{ess}(H) = [0,\infty)$ some mild decay assumptions on V_+ are required). Without further mentioning, similar remarks will apply to the estimates considered below. □

[5] The similarity to estimate (5.6) is obvious. This should explain why (5.6) is usually referred to as a Lieb-Thirring inequality for Jacobi operators.

In recent times, there has been an increasing interest in transferring the Lieb-Thirring inequalities to Schrödinger operators with complex potentials. While the study of such operators certainly does not need to be motivated from a mathematical point of view, let us mention that they also arise in the study of quantum mechanical systems, for instance, in the study of resonances of selfadjoint Schrödinger operators via the method of complex scaling (see, e.g., (Abramov, Aslanyan & Davies 2001)), in certain models of nuclear physics (see (Austern 1967)) and in non-hermitian models of quantum mechanics (see (Bender 2007)).

A first step in transferring the Lieb-Thirring bound to the non-selfadjoint setting has been obtained by Frank, Laptev, Lieb & Seiringer (2006) who considered the eigenvalues of the non-selfadjoint operator $H_0 \dot{+} M_V$ in sectors avoiding the positive half-line.

Theorem 6.2.3. *Let $H = H_0 \dot{+} M_V$ where $V \in L^p(\mathbb{R}^d)$ and $p \geq \frac{d}{2}+1$. Then for every $\chi > 0$ the following holds:*

$$\sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \geq \chi \operatorname{Re}(\lambda)} |\lambda|^{p-\frac{d}{2}} \leq C(p,d)\left(1+\frac{2}{\chi}\right)^p \|(\operatorname{Re}(V))_- + |\operatorname{Im}(V)|\|_{L^p}^p . \tag{6.17}$$

Inequality (6.17) was proven by reduction to a selfadjoint problem, and employing the selfadjoint Lieb-Thirring inequalities.[6] This approach is analogous to the derivation of Theorem 5.1.7 from Theorem 5.1.5 in the context of Jacobi operators. However, we should mention that, contrary to the order of presentation in this thesis, the results on Jacobi operators were inspired by the corresponding results for Schrödinger operators and not vice versa.

In comparison with (6.16) we see that the range of allowed p in (6.17) is more restricted (again, this is similar to the case of Jacobi operators). Moreover, while the sector $\{\lambda : |\operatorname{Im}(\lambda)| \geq \chi \operatorname{Re}(\lambda)\}$ approaches the set $\mathbb{C} \setminus [0,\infty)$ when $\chi \to 0$, the constant on the right-hand side of (6.17) explodes in this limit, suggesting a different behavior of the eigenvalues when approaching $(0,\infty)$ and 0, respectively.

As in the case of Jacobi operators, it is possible to use the inequalities (6.17) to obtain an inequality on the entire discrete spectrum of H.

Corollary 6.2.4 ((Demuth et al. 2009)). *Let $H = H_0 \dot{+} M_V$ where $V \in L^p(\mathbb{R}^d)$ and $p \geq \frac{d}{2}+1$. Then for every $\tau > 0$ the following holds:*

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+\tau}} \leq C(p,d,\tau)\|(\operatorname{Re}(V))_- + |\operatorname{Im}(V)|\|_{L^p}^p. \tag{6.18}$$

Proof. Let us set $W = (\operatorname{Re}(V))_- + |\operatorname{Im}(V)|$. Restricting (6.17) to the set $\operatorname{Re}(\lambda) > 0$, we obtain

$$\sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \geq \chi \operatorname{Re}(\lambda) > 0} |\lambda|^{p-\frac{d}{2}} \leq C(d,p)\left(1+\frac{2}{\chi}\right)^p \|W\|_{L^p}^p. \tag{6.19}$$

[6] See (Bruneau & Ouhabaz 2008) for a generalization of this approach.

Next, we multiply both sides of (6.19) with $\chi^{p-1+\tau}$, where $\tau > 0$, and integrate over $\chi \in (0,1)$. Then for the right-hand side we obtain

$$\int_0^1 d\chi \left(1 + \frac{2}{\chi}\right)^p \chi^{p-1+\tau} \|W\|_{L^p}^p \leq C(d,p,\tau)\|W\|_{L^p}^p. \tag{6.20}$$

Similarly, interchanging sum and integral, we obtain for the left-hand side:

$$\begin{aligned}
&\int_0^1 d\chi\, \chi^{p-1+\tau} \sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \geq \chi \operatorname{Re}(\lambda) > 0} |\lambda|^{p-\frac{d}{2}} \\
= & \sum_{\lambda \in \sigma_d(H), \operatorname{Re}(\lambda) > 0} |\lambda|^{p-\frac{d}{2}} \int_0^{\min(\frac{|\operatorname{Im}(\lambda)|}{\operatorname{Re}(\lambda)},1)} d\chi\, \chi^{p-1+\tau} \\
= & \; C(d,p,\tau) \sum_{\lambda \in \sigma_d(H), \operatorname{Re}(\lambda) > 0} |\lambda|^{p-\frac{d}{2}} \min\left(1, \left(\frac{|\operatorname{Im}(\lambda)|}{\operatorname{Re}(\lambda)}\right)^{p+\tau}\right) \\
\geq & \; C(d,p,\tau) \sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \leq \operatorname{Re}(\lambda)} |\lambda|^{p-\frac{d}{2}} \left(\frac{|\operatorname{Im}(\lambda)|}{\operatorname{Re}(\lambda)}\right)^{p+\tau} \\
\geq & \; C(d,p,\tau) \sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \leq \operatorname{Re}(\lambda)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+\tau}}.
\end{aligned} \tag{6.21}$$

Together, (6.21) and (6.20) show that

$$\sum_{\lambda \in \sigma_d(H), |\operatorname{Im}(\lambda)| \leq \operatorname{Re}(\lambda)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+\tau}} \leq C(d,p,\tau)\|W\|_{L^p}^p.$$

Using Theorem 6.2.3 with $\chi = 1$ gives that the same inequality is true summing over all eigenvalues λ with $|\operatorname{Im}(\lambda)| \geq \operatorname{Re}(\lambda)$. This completes the proof. ■

While providing an estimate on the entire discrete spectrum of H, as compared to the selfadjoint result, Corollary 6.2.4 is still too restrictive with respect to the allowed range of p. One could conjecture that this restriction is necessary, however, the following result due to Laptev & Safronov (2009) indicates that this is not the case.

Theorem 6.2.5. *Let $H = H_0 \dotplus M_V$ where $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ if $d \geq 2$ and $p \geq 1$ if $d = 1$. In addition, let $\operatorname{Re}(V)$ be non-negative and bounded. Then*

$$\sum_{\lambda \in \sigma_d(H)} \left(\frac{|\operatorname{Im}(\lambda)|}{|\lambda+1|^2+1}\right)^p \leq C(p,d)\|\operatorname{Im}(V)\|_{L^p}^p. \tag{6.22}$$

We note that the assumption that $\operatorname{Re}(V)$ is non-negative implies that the spectrum of H is contained in the right half-plane.

Remark 6.2.6. We will not present the proof of inequality (6.22). However, let us mention that it does not rely on complex function theory and is thus completely different from the methods of proof we will apply below. □

In the following, let us compare the inequalities (6.22) and (6.18) given the assumptions of Theorem 6.2.5. To begin, we note that in the numerator of (6.22) we have $|\operatorname{Im}(\lambda)|$ to the power p, whereas in (6.18) a slightly larger exponent $p+\tau$ is required. This shows that (6.22) provides a slightly stronger estimate with respect to convergence to a point in $(0,\infty)$. As another advantage of (6.22) we note that, apart from the case $p = d/2, d \geq 3$, the allowed range of p in (6.22) coincides with the corresponding range in the selfadjoint case. On the other hand, for τ small and p chosen as in Theorem 6.2.5 we have

$$\frac{1}{|\lambda|^{\frac{d}{2}+\tau}} \geq \frac{C(d,p,\tau)}{(|\lambda+1|^2+1)^p}, \qquad \text{if } \operatorname{Re}(\lambda) > 0,$$

which shows that for sequences of eigenvalues converging to 0 and ∞, respectively, inequality (6.18) provides a stronger estimate than inequality (6.22) (at least for $p \geq \frac{d}{2}+1$). In conclusion, we see that neither estimate is strictly better than the other.

In view of the previous discussion, it seems to be natural to ask whether it is possible to combine the advantages of both Corollary 6.2.4 and Theorem 6.2.5, and to derive an estimate of the form

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^p}{|\lambda|^{d/2}} \leq C(p,d)\|(\operatorname{Re}(V))_- + |\operatorname{Im}(V)|\|_{L^p}^p, \tag{6.23}$$

where $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ if $d \geq 2$ and $p \geq 1$ if $d = 1$. While this estimate would obviously be stronger than estimate (6.22), we note that it would also be stronger than estimate (6.18) since for $\lambda \in \mathbb{C} \setminus [0,\infty)$ and $\tau > 0$ we have

$$\frac{\operatorname{dist}(\lambda,[0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+\tau}} \leq \frac{\operatorname{dist}(\lambda,[0,\infty))^p}{|\lambda|^{\frac{d}{2}}},$$

so the left-hand side of (6.23) is larger than the left-hand side of (6.18).

Unfortunately, we cannot provide a complete answer to the question of the validity of (6.23). However, in the next section, by applying the methods developed in Section 3.3, we will at least be able to derive some weaker versions of this inequality, which will combine some of the advantages of Corollary 6.2.4 and Theorem 6.2.5.

6.3. New estimates on the discrete spectrum

Summary: We state and prove new estimates on the discrete spectrum of non-selfadjoint Schrödinger operators and make a comparison with the known estimates discussed in Section 6.2.

6.3.1. Statement of results

In the following, we present estimates on the discrete spectrum of $H = H_0 \dot{+} M_V$ which will be obtained by applying the results established in Section 3.3. Since a crucial step in the proof of these estimates requires the condition $V \in L^p(\mathbb{R}^d)$ with $p \geq 2$, we are not able to improve upon Corollary 6.2.4 and Theorem 6.2.5 if $d \in \{1,2\}$ and we will therefore restrict ourselves to the case $d \geq 3$.

To formulate the next theorem let us recall that $\mathbb{H}^+_\omega = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq \omega\}$.

Theorem 6.3.1. *Let $d \geq 3$ and suppose that $V \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d = 3$ and $p > d/2$ if $d \geq 4$. Furthermore, let $\omega_0 \geq 0$ such that $\mathrm{Num}(H) \subset \mathbb{H}^+_{-\omega_0}$. Then for $\tau \in (0,1)$ the following holds:*

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}^c} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \quad \leq \quad C \|V\|^p_{L^p}. \tag{6.24}$$

Moreover, if $p \geq d - \tau$ then

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{d/2}} \quad \leq \quad C \|V\|^p_{L^p}, \tag{6.25}$$

and if $p < d - \tau$ then

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}} (|\lambda| + \omega_0)^{\frac{d-p-\tau}{2}}} \quad \leq \quad C \|V\|^p_{L^p}. \tag{6.26}$$

In each case, $C = C(p, d, \omega_0, \tau)$.

Remark 6.3.2. More precisely, in all but one case we have $C = C(p,d,\tau)(1+\omega_0)^\tau$, the only exception being inequality (6.24) where $C = C(p,d,\tau)(1+\omega_0)^{\frac{d-p+\tau}{2}}$ if $p < d - \tau$. □

Remark 6.3.3. In view of Corollary 6.2.4 we know that estimate (6.25) is actually valid for every $p \geq \frac{d}{2} + 1$ (note that $d > \frac{d}{2} + 1$ since $d \geq 3$). However, we prefer to present the theorem as stated above, since this is what we obtain by applying the results developed in Section 3.3. □

As in Remark 3.3.10 we note that the choice of the unit disk $\mathbb{D}$ in the above estimates is not essential and it can be replaced with any disk centered at zero by a suitable adaption of the constants on the right-hand sides. In view of this fact, let us compare the above theorem with Corollary 6.2.4 assuming that $\omega_0 > 0$ (the case $\omega_0 = 0$ will be considered separately below). In this case, both Theorem 6.3.1 and Corollary 6.2.4 provide similar estimates on sequences converging to $(0,\infty)$, namely, for $V \in L^p(\mathbb{R}^d)$ they imply that (ignoring the different factors in front of τ)

$$\sum_{\lambda \in \sigma_d(H), |\lambda| > \varepsilon} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} < \infty, \qquad \varepsilon > 0.$$

However, while Corollary 6.2.4 does allow this conclusion for $p \geq \frac{d}{2} + 1$ only, the previous theorem implies the finiteness of this sum for every $p > d/2$ if $d \geq 4$ and $p \geq 2$ if $d = 3$.

As we have already implicitly stated in Remark 6.3.3, regarding sequences converging to zero Corollary 6.2.4 provides stronger estimates than the previous theorem if $d > p \geq \frac{d}{2}+1$ (since $(p+\tau)/2 < d/2+\tau$ in this case), and both results provide similar estimates in case $p \geq d$. On the other hand, while the corollary does not provide any estimates in case that $p < d/2+1$, estimate (6.26) remains valid for $p \in (\frac{d}{2}, \frac{d}{2}+1)$ if $d \geq 4$ and $p \in [2, \frac{5}{2})$ if $d = 3$. Whether the different structure of the estimates (6.25) and (6.26) does indeed reflect an essential difference in the behavior of the discrete spectrum of H for $V \in L^p(\mathbb{R}^d)$ with $p \in (\frac{d}{2}, \frac{d}{2}+1)$, or is rather an artifact of our method of proof, remains an open problem.

Finishing our comparison of Corollary 6.2.4 and the above theorem, let us note that in contrast to the right-hand side in (6.18), the right-hand sides of (6.24)-(6.26) do depend on the unknown parameter ω_0 (which, in principle, could be estimated in terms of V) and on the positive part of $\mathrm{Re}(V)$.

Next, let us consider the case when ω_0 can be chosen as 0 (which is only possible if the spectrum of H is contained in the right half-plane). In this case, the previous theorem immediately implies the following corollary.

Corollary 6.3.4. *Let $d \geq 3$ and suppose that $V \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d = 3$ and $p > d/2$ if $d \geq 4$. In addition, suppose that $\mathrm{Num}(H) \subset \mathbb{H}_0^+$. Then for $\tau \in (0,1)$ the following holds:*

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{d/2}} + \sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}^c} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \quad \leq \quad C(p,d,\tau)\|V\|_{L^p}^p. \quad (6.27)$$

Remark 6.3.5. In particular, $\mathrm{Num}(H) \subset \mathbb{H}_0^+$ if $\mathrm{Re}(V) \geq 0$. □

We see that inequality (6.27) is pretty close to the "desired" inequality (6.23) mentioned at the end of the previous section. Indeed, setting $\tau = 0$, and ignoring the fact that the right-hand side of (6.23) does not depend on $(\mathrm{Re}(V))_+$, these inequalities actually coincide. In particular, we see that in contrast to Theorem 6.3.1 the previous corollary provides the same estimates on sequences of eigenvalues converging to 0, independent of the fact whether $p \in (\frac{d}{2}, \frac{d}{2}+1)$ or $p \geq \frac{d}{2}+1$. Once again, we don't know whether this is an essential feature, related to the fact that in the previous corollary we only consider eigenvalues in the right-half plane, or whether this difference is due to our method of proof.

Finally, as compared to Theorem 6.2.5 let us note that the previous corollary provides similar estimates on sequences accumulating on $(0,\infty)$ (ignoring the τ-corrections), and stronger estimates on sequences converging to 0.

Remark 6.3.6. All estimates presented in this chapter provide bounds in terms of the L^p-norm of V. While this condition is indeed required to apply the results of Section 3.3, it is natural to conjecture that it is possible to replace the L^p-norm of V with the L^p-norm of $((\mathrm{Re}(V))_- + |\mathrm{Im}(\lambda)|$ without changing the results. □

To conclude, while we have seen that each of the estimates presented in the present and the previous chapter has certain advantages and disadvantages, it remains an open question whether it is indeed possible to combine the advantages of all these estimates and to obtain an estimate of the form, or close to the form, of inequality (6.23).

6.3.2. Proof of Theorem 6.3.1.

We begin with a useful estimate.

Lemma 6.3.7. *For $\lambda \in \mathbb{C} \setminus [0,\infty)$ we have*

$$|\lambda|^{1/2}|\operatorname{Im}(\sqrt{\lambda})| \leq \operatorname{dist}(\lambda,[0,\infty)) \leq 2|\lambda|^{1/2}|\operatorname{Im}(\sqrt{\lambda})|.$$

Proof. Let $\mu = \sqrt{\lambda}$, where the square root is chosen such that $\operatorname{Im}(\mu) > 0$. If $\operatorname{Re}(\mu^2) > 0$ then $|\operatorname{Re}(\mu)| > |\operatorname{Im}(\mu)|$ and we have

$$\operatorname{dist}(\mu^2,[0,\infty)) = |\operatorname{Im}(\mu^2)| = 2|\operatorname{Re}(\mu)||\operatorname{Im}(\mu)| \leq 2|\mu||\operatorname{Im}(\mu)|,$$

$$\operatorname{dist}(\mu^2,[0,\infty)) = |\operatorname{Im}(\mu^2)| = 2|\operatorname{Re}(\mu)||\operatorname{Im}(\mu)| \geq \sqrt{2}|\mu||\operatorname{Im}(\mu)|.$$

If $\operatorname{Re}(\mu^2) \leq 0$ then $|\operatorname{Re}(\mu)| \leq |\operatorname{Im}(\mu)|$ and we have

$$\operatorname{dist}(\mu^2,[0,\infty)) = |\mu|^2 \leq 2\operatorname{Im}(\mu)^2 \leq 2|\operatorname{Im}(\mu)||\mu|,$$

$$\operatorname{dist}(\mu^2,[0,\infty)) = |\mu|^2 \geq |\mu||\operatorname{Im}(\mu)|.$$

Taking the worst-case scenarios we get the result. ∎

We intend to prove Theorem 6.3.1 using Theorem 3.3.9. Hence, the proof will rely on an estimate on the $\mathcal{S}_p$-norm of $M_V R_{H_0}(\lambda)$ (of course, up to this point it is not even clear that this operator is bounded on $L^2(\mathbb{R}^d)$). Since $R_{H_0}(\lambda) = F^{-1}M_{k_\lambda}F$, where

$$k_\lambda(x) = (\lambda - |x|^2)^{-1}, \quad x \in \mathbb{R}^d, \quad \lambda \in \mathbb{C} \setminus [0,\infty), \tag{6.28}$$

as in the case of Jacobi operators this estimate can be reduced to an estimate on the L^p-norm of the bounded function k_λ.

Lemma 6.3.8. *Let $d \geq 1$. Then for $\lambda \in \mathbb{C} \setminus [0,\infty)$ and k_λ as defined in (6.28) the following holds: If $p > \max(d/2,1)$ then*[7]

$$\|k_\lambda\|^p_{L^p(\mathbb{R}^d)} \leq C(p,d)\frac{|\lambda|^{\frac{d}{2}-1}}{\operatorname{dist}(\lambda,[0,\infty))^{p-1}}. \tag{6.29}$$

Remark 6.3.9. While we don't need the above estimates in case $d \in \{1,2\}$ for the proof of Theorem 6.3.1, we do present them since we still need to show that $M_W R_{H_0}(\lambda)$ is compact if $W \in C_0^\infty(\mathbb{R}^d), d \geq 1$, to complete the proof of Proposition 6.1.8. □

[7] Here we deviate from our standard notation and write $\|.\|_{L^p(\mathbb{R}^d)}$ instead of $\|.\|_{L^p}$.

Proof. To begin, let us assume that $\mathrm{Re}(\lambda) \leq 0$. Then we obtain

$$\begin{aligned}
\|k_\lambda\|^p_{L^p(\mathbb{R}^d)} &= C(d) \int_0^\infty \frac{r^{d-1}\, dr}{((\mathrm{Re}(\lambda) - r^2)^2 + \mathrm{Im}(\lambda)^2)^{p/2}} \\
&\leq C(d) \int_0^\infty \frac{r^{d-1}\, dr}{(r^4 + |\lambda|^2)^{p/2}} = \frac{C(d)}{|\lambda|^{p-\frac{d}{2}}} \int_0^\infty \frac{s^{d-1}\, ds}{(s^4+1)^{p/2}} = \frac{C(p,d)}{|\lambda|^{p-\frac{d}{2}}},
\end{aligned}$$

the last integral being finite since $p > d/2$. Since $\mathrm{dist}(\lambda, [0,\infty)) = |\lambda|$ if $\mathrm{Re}(\lambda) \leq 0$, we obtain (6.29).

In the following, we consider the case $\mathrm{Re}(\lambda) > 0$. Here, it is convenient to distinguish between the cases $d = 1$ and $d \geq 2$, respectively. We start with a consideration of the case $d \geq 2$. To this end, let us denote $\lambda_0 = \mathrm{Re}(\lambda)$ and $\lambda_1 = \mathrm{Im}(\lambda)$. Since $\|k_\lambda\|_{L^p(\mathbb{R}^d)} = \|k_{\overline{\lambda}}\|_{L^p(\mathbb{R}^d)}$, it is sufficient to treat the case $\lambda_1 > 0$. With the change of variable $r = \sqrt{\lambda_0 - \lambda_1 s}$ we obtain

$$\begin{aligned}
\|k_\lambda\|^p_{L^p(\mathbb{R}^d)} &= C(d) \int_0^\infty \frac{r^{d-1}\, dr}{((\lambda_0 - r^2)^2 + \lambda_1^2)^{p/2}} = C(d)\lambda_1^{1-p} \int_{-\infty}^{\lambda_0/\lambda_1} \frac{(\lambda_0 - \lambda_1 s)^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} \\
&= C(d)\lambda_1^{1-p} \left[\int_0^\infty \frac{(\lambda_0 + \lambda_1 s)^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} + \int_0^{\lambda_0/\lambda_1} \frac{(\lambda_0 - \lambda_1 s)^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} \right]. \quad (6.30)
\end{aligned}$$

For the first integral on the right-hand side of (6.30), we have, using that $d \geq 2$ and so $(\lambda_0 + \lambda_1 s)^{(d-2)/2} \leq (2\lambda_0)^{(d-2)/2} + (2\lambda_1 s)^{(d-2)/2}$,

$$\begin{aligned}
\int_0^\infty \frac{(\lambda_0 + \lambda_1 s)^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} &\leq C(d) \left[\lambda_0^{(d-2)/2} \int_0^\infty \frac{ds}{(s^2+1)^{p/2}} + \lambda_1^{(d-2)/2} \int_0^\infty \frac{s^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} \right] \\
&\leq C(d,p)[\lambda_0^{(d-2)/2} + \lambda_1^{(d-2)/2}], \quad (6.31)
\end{aligned}$$

where the integrals are finite by the assumption that $p > d/2$ and $d \geq 2$. For the second integral on the right-hand side of (6.30) we have, using again that $p > 1$,

$$\begin{aligned}
\int_0^{\lambda_0/\lambda_1} \frac{(\lambda_0 - \lambda_1 s)^{(d-2)/2}\, ds}{(s^2+1)^{p/2}} &\leq \lambda_0^{(d-2)/2} \int_0^{\lambda_0/\lambda_1} \frac{ds}{(s^2+1)^{p/2}} \quad (6.32) \\
&\leq \lambda_0^{(d-2)/2} \int_0^\infty \frac{ds}{(s^2+1)^{p/2}} = C(p)\lambda_0^{(d-2)/2}.
\end{aligned}$$

The inequalities (6.30), (6.31) and (6.32) imply that, with $\lambda_0 = \mathrm{Re}(\lambda) > 0$ and $\lambda_1 = \mathrm{Im}(\lambda)$,

$$\|k_\lambda\|^p_{L^p(\mathbb{R}^d)} \leq C(d,p) \left[\frac{|\lambda_0|^{(d-2)/2}}{|\lambda_1|^{p-1}} + \frac{1}{|\lambda_1|^{p-\frac{d}{2}}} \right] \leq C(d,p) \frac{|\lambda|^{\frac{d}{2}-1}}{\mathrm{dist}(\lambda, [0,\infty))^{p-1}},$$

where in the last inequality we have used that $|\lambda_1| = \mathrm{dist}(\lambda, [0,\infty))$ and $|\lambda_0|, |\lambda_1| \leq |\lambda|$, respectively. This concludes the proof of (6.29) in case that $d \geq 2$.

It remains to consider the case $d = 1$ and $\mathrm{Re}(\lambda) > 0$: with $\mu = \sqrt{\lambda}$, where $\mathrm{Im}(\mu) > 0$, we obtain

$$\|k_\lambda\|^p_{L^p(\mathbb{R})} = 2 \int_0^\infty \frac{dr}{|\mu - r|^p |\mu + r|^p}. \quad (6.33)$$

We denote $\mathrm{Re}(\mu) = \mu_0$ and $\mathrm{Im}(\mu) = \mu_1$. To begin, let us assume that $\mu_0 > 0$. Then $|\mu+r| \geq |\mu|$, where $r > 0$, and we obtain

$$\|k_\lambda\|^p_{L^p(\mathbb{R})} \leq \frac{2}{|\mu|^p} \int_0^\infty \frac{dr}{|\mu - r|^p} = \frac{2}{|\mu|^p} \int_0^\infty \frac{dr}{((\mu_0 - r)^2 + \mu_1^2)^{p/2}}. \tag{6.34}$$

Making a change of variables $s = \frac{\mu_0 - r}{\mu_1}$, we have

$$\int_0^\infty \frac{dr}{((\mu_0 - r)^2 + \mu_1^2)^{p/2}} = \frac{1}{\mu_1^{p-1}} \int_{-\infty}^{\mu_0/\mu_1} \frac{ds}{(s^2+1)^{p/2}} \leq \frac{1}{\mu_1^{p-1}} \int_{-\infty}^{\infty} \frac{ds}{(s^2+1)^{p/2}}, \tag{6.35}$$

where the integral on the right-hand side is finite since $p > 1$. From (6.34) and (6.35) we obtain for $\mu_0 = \mathrm{Re}(\sqrt{\lambda}) > 0$

$$\|k_\lambda\|^p_{L^p(\mathbb{R})} \leq \frac{2}{|\lambda|^{p/2}\, \mathrm{Im}(\sqrt{\lambda})^{p-1}} \leq \frac{2^p}{|\lambda|^{1/2}\, \mathrm{dist}(\lambda, [0,\infty))^{p-1}}, \tag{6.36}$$

where in the last inequality we used Lemma 6.3.7. For $\mu_0 = \mathrm{Re}(\sqrt{\lambda}) \leq 0$, we just note that, as above,

$$\|k_\lambda\|^p_{L^p(\mathbb{R})} \leq \frac{2}{|\mu|^p} \int_0^\infty \frac{dr}{((\mu_0 + r)^2 + \mu_1^2)^{p/2}} = \frac{2}{|\mu|^p} \int_0^\infty \frac{dr}{(((-\mu_0) - r)^2 + \mu_1^2)^{p/2}},$$

and (6.36) follows immediately from the previous computations. This concludes the proof of (6.29). ■

With the previous lemma, we can now provide a bound on the $\mathcal{S}_p$-norm of $M_V R_{H_0}(\lambda)$.

Lemma 6.3.10. *Let $d \geq 1$ and $V \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d \geq 4$. Then for $\lambda \in \mathbb{C} \setminus [0,\infty)$ we have*

$$\|M_V R_{H_0}(\lambda)\|^p_{\mathcal{S}_p} \leq C(p,d) \|V\|^p_{L^p} \frac{|\lambda|^{\frac{d}{2}-1}}{\mathrm{dist}(\lambda,[0,\infty))^{p-1}}. \tag{6.37}$$

Remark 6.3.11. In particular, $M_W R_{H_0}(\lambda) \in \mathcal{S}_\infty$ if $W \in C_0^\infty(\mathbb{R}^d)$. □

Proof. With k_λ as defined in (6.28) we obtain

$$\|M_V[\lambda - H_0]^{-1}\|^p_{\mathcal{S}_p} = \|M_V F^{-1} M_{k_\lambda} F\|^p_{\mathcal{S}_p} \leq \|M_{|V|} F^{-1} M_{|k_\lambda|} F\|^p_{\mathcal{S}_p},$$

where we used the unitarity of $M_{V/|V|}$ and $M_{k_\lambda/|k_\lambda|}$, respectively. Since $p \geq 2$ by assumption, using Theorem 4.1 in (Simon 2005) (compare with Lemma 5.2.8 above), the right-hand side of the last inequality can be bounded from above by

$$(2\pi)^{-d/(2p)} \|k_\lambda\|^p_{L^p} \|V\|^p_{L^p}.$$

But then an application of Lemma 6.3.8 concludes the proof. ■

Remark 6.3.12. The above lemma, or rather its proof, is the reason for our restriction to $p \geq 2$ in the formulation of Theorem 6.3.1. If, similar to the case of Jacobi operators, the proof of Theorem 6.3.1 could be reduced to an estimate on $\|M_{|V|^{1/2}} R_{H_0}(\lambda) M_{|V|^{1/2}}\|_{\mathcal{S}_p}^p$, then this restriction could be overcome. $\square$

We are finally prepared for the proof of Theorem 6.3.1. In the following, let us assume that $V \in L^p(\mathbb{R}^d)$, where $d \geq 3$, and that $p \geq 2$ if $d = 3$ and $p > d/2$ if $d \geq 4$. Given this assumption, the previous lemma shows that the operator M_V is relatively compact with respect to H_0. In particular, in view of Lemma 6.1.4 and Lemma 3.3.4 we see that the operator $H_0 + M_V$ coincides with form sum $H = H_0 \dot{+} M_V$.

Let us recall that by assumption we have $\mathrm{Num}(H) \subset \mathbb{H}^+_{-\omega_0}$, where $\omega_0 \geq 0$, so in view of Remark 6.1.5 we obtain $\mathbb{H}^-_{-\omega_0} = \{\lambda : \mathrm{Re}(\lambda) < -\omega_0\} \subset \rho(H)$ and

$$\|R_H(\lambda)\| \leq |\operatorname{Re}(\lambda) + \omega_0|^{-1}, \quad \lambda \in \mathbb{H}^-_{-\omega_0}. \tag{6.38}$$

Hence, with the terminology of Section 3.3 (see (3.44)) we have $-\omega_0 \in \mathbb{L}(H)$. Next, we note that by Lemma 6.3.10 we have

$$\|M_V R_{H_0}(\lambda)\|_{\mathcal{S}_p}^p \leq \frac{C(p,d)\|V\|_{L^p}^p |\lambda|^{\frac{d}{2}-1}}{\operatorname{dist}(\lambda, [0,\infty))^{p-1}}, \qquad \lambda \in \mathbb{C} \setminus [0,\infty). \tag{6.39}$$

The inequalities (6.39) and (6.38) show that we can apply Theorem 3.3.9 with $C_0 = 1$, $\alpha = p-1$, $\beta = \frac{d}{2}-1$ and $K = C(p,d)\|V\|_{L^p}^p$. Hence, for $\tau \in (0,1)$ we obtain $\eta_0 = \frac{d}{2}-p-\tau$, $\eta_1 = p+\tau$ and $\eta_2 = p-d+\tau$ if $p \geq d-\tau$ and $\eta_2 = 0$ if $p < d-\tau$. If $p \geq d-\tau$, then inserting these parameters into (3.56) and (3.57) we obtain

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{d/2}} \leq C(p,d,\tau)(1+\omega_0)^\tau \|V\|_{L^p}^p$$

and

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}^c} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \leq C(p,d,\tau)(1+\omega_0)^\tau \|V\|_{L^p}^p.$$

Similarly, if $p < d-\tau$ then we obtain

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}}(|\lambda|+\omega_0)^{\frac{d-p-\tau}{2}}} \leq C(p,d,\tau)(1+\omega_0)^\tau \|V\|_{L^p}^p$$

and

$$\sum_{\lambda \in \sigma_d(H) \cap \mathbb{D}^c} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \leq C(p,d,\tau)(1+\omega_0)^{\frac{d-p+\tau}{2}} \|V\|_{L^p}^p.$$

This concludes the proof of Theorem 6.3.1. $\blacksquare$

Appendices

A. Harmonic and subharmonic functions

We present some standard results about harmonic and subharmonic functions, mainly restricting ourselves to functions defined on the unit disk. We refer to (Ransford 1995) for a deeper discussion of these topics.

In the following, U and Ω denote bounded regions[8] in the complex plane, where, in addition, Ω is simply connected.

A.1. Harmonic functions

A function $u : U \to \mathbb{R}$ is called **harmonic** if $u \in C^2(U)$ and $\Delta u := (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = 0$ on U. For instance, the Cauchy-Riemann equations show that the real part of a holomorphic function is harmonic. On simply connected regions also the opposite is true, i.e., if g is harmonic on Ω then there exists a function $f \in H(\Omega)$ such that $g = \mathrm{Re}(f)$. In particular, the function $g = \log |f|$ is harmonic on U whenever $f \in H(U)$ and $f \neq 0$ on U.

Harmonic functions satisfy the **mean value property**: If u is harmonic on $\mathbb{D}_R(z_0) = \{z : |z - z_0| < R\}$ then for every $r \in (0, R)$ we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta. \tag{A.1}$$

The mean-value property implies the following **maximum principle**: If $u : U \to \mathbb{R}$ is harmonic and attains a local maximum on U, then u is constant. Furthermore, if u extends continuously onto $\overline{U}$ and $u \leq 0$ on ∂U, then $u \leq 0$ on U.

The **Dirichlet problem** for a region U consists in finding a harmonic function $u : U \to \mathbb{R}$ which satisfies $\lim_{z \to \xi} u(z) = \varphi(\xi)$ for every $\xi \in \partial U$, where φ is a given continuous function defined on the boundary of U. For the unit disk $\mathbb{D}$ the Dirichlet problem is solved by means of the **Poisson integral**: Let $\varphi : \mathbb{T} \to \mathbb{R}$ be integrable and define $P_{\mathbb{D}}\varphi : \mathbb{D} \to \mathbb{R}$ via

$$P_{\mathbb{D}}\varphi(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \varphi(e^{i\theta}) d\theta. \tag{A.2}$$

Since $P_{\mathbb{D}}\varphi$ can be expressed as the real part of a holomorphic function, it is harmonic on $\mathbb{D}$. Furthermore, if φ is continuous at $\xi_0 \in \mathbb{T}$ then $\lim_{w \to \xi_0} P_{\mathbb{D}}\varphi(w) = \varphi(\xi_0)$.

[8] A set $U \subset \mathbb{C}$ is called a region if it is open and connected.

For $\theta_1 < \theta_2$ let $J = [e^{i\theta_1}, e^{i\theta_2}]$ be a closed arc on the unit circle and let χ_J denote the corresponding characteristic function. The harmonic function $\omega_J = P_{\mathbb{D}}\chi_J : \mathbb{D} \to \mathbb{R}$ is called the **harmonic measure** of J with respect to $\mathbb{D}$. It is explicitly given by

$$\omega_J(w) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \frac{1-|w|^2}{|e^{i\theta}-w|^2} d\theta, \tag{A.3}$$

and we have $\lim_{w\to\xi}\omega_J(w) = \chi_J(\xi)$ for every $\xi \in \mathbb{T}\setminus\{e^{i\theta_1}, e^{i\theta_2}\}$.

The formula above can be used to derive the following geometric interpretation of the harmonic measure:

$$\omega_J(w) = \frac{1}{\pi}\left(\theta(w) - \frac{\theta_2-\theta_1}{2}\right) \tag{A.4}$$

where $J = [e^{i\theta_1}, e^{i\theta_2}]$ is seen from w under the angle $\theta(w)$, see (Garnett 1981) Chapter 1, Exercise 3.

A.2. Subharmonic functions

A function $v : U \to [-\infty, \infty)$ is called **subharmonic** if it is upper semicontinuous and satisfies the **local sub-mean inequality**, i.e., for every $z \in U$ there exists $R = R(z) > 0$ such that

$$v(z) \le \frac{1}{2\pi}\int_0^{2\pi} v(z + re^{i\theta})d\theta \tag{A.5}$$

for every $r \in (0, R)$. Since upper semicontinuous functions are bounded above on compact sets, the above integral is well defined (although it can be equal to $-\infty$).

Clearly, harmonic functions are subharmonic. More importantly, if $f \in H(U)$ then $v = \log|f|$ is subharmonic on U. The following version[9] of the maximum principle is due to Lindelöf: Let v be subharmonic and bounded above on U. If F is a finite subset of ∂U and $\limsup_{z\to\xi} u(z) \le 0$ for all $\xi \in \partial U \setminus F$, then $u \le 0$ on U. In particular, this result also generalizes the maximum principle for harmonic functions mentioned above.

If $v \in C^2(U)$ then v is subharmonic if and only if $\Delta v \ge 0$. In this case, using Green's identity it can be shown that the linear functional

$$\Lambda_v : \varphi \mapsto \int_U v(z)\Delta\varphi(z)dz, \quad \varphi \in C_0^\infty(U), \tag{A.6}$$

is positive, i.e., $\varphi \ge 0$ implies that $\Lambda_v\varphi \ge 0$ as well. For general subharmonic functions v, with $v \not\equiv -\infty$, the functional Λ_v is still well defined since v is locally integrable. An approximation argument shows that it remains positive as well. The unique Radon measure corresponding to that functional via the Riesz representation theorem will be denoted by Δ_v, that is, for every $\varphi \in C_0^\infty(U)$ we have

$$\int_U v(z)\Delta\varphi(z)dz = \int_U \varphi(z)\Delta_v(dz). \tag{A.7}$$

[9] See, e.g., (Garnett & Marshall 2005), Exercise 3 on p.27.

The measure Δ_v is called the **Riesz measure** of v. Most importantly, if $v = \log|f|$ for some (non-trivial) holomorphic function $f \in H(U)$, then $\frac{1}{2\pi}\Delta_v$ is a discrete measure supported on the zero set $\mathcal{Z}(f)$ and $\frac{1}{2\pi}\Delta_v(\{z\})$ coincides with the order of z as a zero of f.

Let $v : U \to [-\infty, \infty)$ be subharmonic. A harmonic function $u : U \to \mathbb{R}$ is called a **harmonic majorant** of v on U if $u \geq v$ on U. If $u \leq k$ for every other harmonic majorant k of v, then u is called the **least harmonic majorant** of v on U.

Suppose that v is subharmonic on Ω and $v \not\equiv -\infty$. If v has a harmonic majorant on Ω, then it has a least one, we call it u, and

$$v(w) = u(w) - \frac{1}{2\pi}\int_\Omega G_\Omega(w,z)\Delta_v(dz), \quad w \in \Omega, \tag{A.8}$$

where $G_\Omega : \Omega \times \Omega \to (-\infty, \infty]$ denotes the **Green's function** of that region, that is, the unique function with

(i) $G_\Omega(\,.\,,z)$ is harmonic on $\Omega \setminus \{z\}$ and bounded outside each neighborhood of z,

(ii) $G_\Omega(w,w) = \infty$ and $G_\Omega(w,z) = -\log|z-w| + O(1)$ as $w \to z$,

(iii) $G_\Omega(w,z) \to 0$ as $w \to \xi \in \partial\Omega$.

For example, the Green's function of the unit disk $\mathbb{D}$ is given by

$$G_{\mathbb{D}}(w,z) = \log\left|\frac{1-w\overline{z}}{w-z}\right|. \tag{A.9}$$

B. Sectorial forms and operators

We discuss the connection between sectorial forms and sectorial operators. As a primary reference and for a much more detailed treatment of this topic we refer to the monograph (Kato 1995). Throughout this section, we consider linear operators acting in a complex separable Hilbert space $\mathcal{H}$. The inner product on $\mathcal{H}$, which is linear in the first and semilinear in the second component, is denoted by $\langle\,.\,,.\rangle$.

B.1. Sectorial operators

The **numerical range** of a linear operator Z in $\mathcal{H}$ is defined as

$$\mathrm{Num}(Z) = \{\langle Zf, f\rangle \in \mathbb{C} : f \in \mathrm{Dom}(Z), \|f\| = 1\}. \tag{B.1}$$

A classical result of Toeplitz and Hausdorff says that $\mathrm{Num}(Z)$, and hence its closure, is convex. In particular, $\mathbb{C}\setminus\overline{\mathrm{Num}}(Z)$ is connected, except in the special case when $\overline{\mathrm{Num}}(Z)$ is a strip bounded by two parallel lines. For a proof of the following theorem we refer to (Kato 1995), p.268.

Theorem B.1. *Let Z be a closed operator in $\mathcal{H}$ and assume that $\Delta = \mathbb{C} \setminus \overline{\mathrm{Num}}(Z)$ is connected. If $\rho(Z) \cap \Delta \neq \emptyset$ then $\Delta \subset \rho(Z)$, that is, $\sigma(Z) \subset \overline{\mathrm{Num}}(Z)$, and*

$$\|R_Z(\lambda)\| \leq \frac{1}{\mathrm{dist}(\lambda, \overline{\mathrm{Num}}(Z))}, \quad \lambda \in \Delta. \tag{B.2}$$

An operator Z in $\mathcal{H}$ is called **accretive** if its numerical range is contained in the right half-plane, i.e., for all $f \in \mathrm{Dom}(Z)$ we have

$$\mathrm{Re}\langle Zf, f\rangle \geq 0. \tag{B.3}$$

If $\{\lambda : \mathrm{Re}(\lambda) < 0\} \subset \rho(Z)$ and

$$\|R_Z(\lambda)\| \leq \frac{1}{|\,\mathrm{Re}(\lambda)|} \quad \text{for } \mathrm{Re}(\lambda) < 0, \tag{B.4}$$

then Z is said to be **m-accretive**. We note that an m-accretive operator is maximal accretive, that is, it is accretive and it has no proper accretive extension. Moreover, every m-accretive operator is closed and densely defined. Z is called **quasi-accretive** (**quasi-m-accretive**) if $Z + a$ is accretive (m-accretive) for some $a \in \mathbb{R}$.

A quasi-accretive operator Z is called **sectorial** if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ such that

$$\mathrm{Num}(Z) \subset \{\lambda : |\arg(\lambda - \gamma)| \leq \theta\} =: S_{\gamma,\theta}. \tag{B.5}$$

Here γ and θ are called a **vertex** and a **semi-angle** of Z, respectively (of course, both are not unique). If Z is sectorial and quasi-m-accretive, then Z is called **m-sectorial**.

Example B.2. If Z is selfadjoint and non-negative, then Z is m-sectorial with vertex and semi-angle 0. □

Let Z be m-sectorial with vertex γ and semi-angle θ. Since Z is quasi-m-accretive, the set $\rho(Z) \cap \mathbb{C} \setminus \overline{\mathrm{Num}(Z)}$ is non-empty. Therefore, we obtain from Theorem B.1 that

$$\sigma(Z) \subset \overline{\mathrm{Num}}(Z) \subset S_{\gamma,\theta} \tag{B.6}$$

and

$$\|R_Z(\lambda)\| \leq \frac{1}{\mathrm{dist}(\lambda, \overline{\mathrm{Num}}(Z))} \leq \frac{1}{\mathrm{dist}(\lambda, S_{\gamma,\theta})} \tag{B.7}$$

for $\lambda \notin S_{\gamma,\theta}$.

B.2. Sectorial forms

Throughout this section, $\mathcal{E} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ denotes a (sesquilinear) **form** in $\mathcal{H}$, that is, $\mathcal{E}(u, v)$ is linear in the first and semilinear in the second component. The domain of $\mathcal{E}$ is denoted by $\mathrm{Dom}(\mathcal{E}) \subset \mathcal{H}$. For $u \in \mathrm{Dom}(\mathcal{E})$ we define the associated **quadratic form** by $\mathcal{E}[u] = \mathcal{E}(u, u)$. If $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ are forms in $\mathcal{H}$ and $\alpha \in \mathbb{C}$, then $\alpha\mathcal{E}$, with $\mathrm{Dom}(\alpha\mathcal{E}) =$

$\mathrm{Dom}(\mathcal{E})$, and $\mathcal{E}_1 + \mathcal{E}_2$, with $\mathrm{Dom}(\mathcal{E}_1 + \mathcal{E}_2) = \mathrm{Dom}(\mathcal{E}_1) \cap \mathrm{Dom}(\mathcal{E}_2)$, are defined in the obvious way. Moreover, the adjoint form $\mathcal{E}^*$ is defined by

$$\mathcal{E}^*(u,v) = \overline{\mathcal{E}(v,u)}, \quad \mathrm{Dom}(\mathcal{E}^*) = \mathrm{Dom}(\mathcal{E}). \tag{B.8}$$

The form $\mathcal{E}$ is called **symmetric** if $\mathcal{E} = \mathcal{E}^*$. We set

$$\mathrm{Re}(\mathcal{E}) = \frac{1}{2}(\mathcal{E} + \mathcal{E}^*), \quad \mathrm{Im}(\mathcal{E}) = \frac{1}{2i}(\mathcal{E} - \mathcal{E}^*). \tag{B.9}$$

Clearly, $\mathrm{Re}(\mathcal{E})$ and $\mathrm{Im}(\mathcal{E})$ are symmetric with

$$\mathrm{Re}(\mathcal{E}[u]) = (\mathrm{Re}(\mathcal{E}))[u] \quad \text{and} \quad \mathrm{Im}(\mathcal{E}[u]) = (\mathrm{Im}(\mathcal{E}))[u].$$

The **numerical range** of $\mathcal{E}$ is defined as

$$\mathrm{Num}(\mathcal{E}) = \{\mathcal{E}[u] \in \mathbb{C} : u \in \mathrm{Dom}(\mathcal{E}), \|u\| = 1\}. \tag{B.10}$$

As in the case of operators, $\mathrm{Num}(\mathcal{E})$ is a convex set. The form $\mathcal{E}$ will be called **sectorial** if this set is contained in a sector, i.e.,

$$\mathrm{Num}(\mathcal{E}) \subset S_{\gamma,\theta} \tag{B.11}$$

where $\gamma \in \mathbb{R}, \theta \in [0, \pi/2)$ and $S_{\gamma,\theta}$ was defined in (B.5). Again, γ and θ are called a vertex and a semi-angle of $\mathcal{E}$, respectively.

In the following, let $\mathcal{E}$ be sectorial. A sequence $u_n \in \mathrm{Dom}(\mathcal{E})$ is called **$\mathcal{E}$-convergent**, in symbol $u_n \overset{\mathcal{E}}{\to} u$, if $u_n \to u$ and $\mathcal{E}[u_n - u_m] \to 0$ for $n, m \to \infty$. We say that $\mathcal{E}$ is **closed**, if $u_n \overset{\mathcal{E}}{\to} u$ implies that $u \in \mathrm{Dom}(\mathcal{E})$ and $\mathcal{E}[u_n - u] \to 0$. One can show that $\mathcal{E}$ is closed if and only if $\mathrm{Re}(\mathcal{E})$ is closed.

The sectorial form $\mathcal{E}$ is called **closable** if it has a closed extension. This is the case if and only if $u_n \overset{\mathcal{E}}{\to} 0$ implies that $\mathcal{E}[u_n] \to 0$. The smallest closed extension $\tilde{\mathcal{E}}$ of $\mathcal{E}$ is then defined as follows: $\mathrm{Dom}(\tilde{\mathcal{E}})$ consists of all $u \in \mathcal{H}$ such that there exists a sequence $\{u_n\}$ with $u_n \overset{\mathcal{E}}{\to} u$, and

$$\tilde{\mathcal{E}}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n, v_n) \text{ for any } u_n \overset{\mathcal{E}}{\to} u, v_n \overset{\mathcal{E}}{\to} v.$$

Example B.3. If Z is a sectorial operator in $\mathcal{H}$ with vertex γ and semi-angle θ, then

$$\mathcal{E}_Z(u,v) = \langle Zu, v\rangle, \quad \mathrm{Dom}(\mathcal{E}_Z) = \mathrm{Dom}(Z),$$

defines a closable sectorial form with vertex γ and semi-angle θ. □

A form $\mathcal{E}$ is called relatively bounded with respect to a sectorial form $\mathcal{E}_0$ (or simply $\mathcal{E}_0$-bounded), if $\mathrm{Dom}(\mathcal{E}) \supset \mathrm{Dom}(\mathcal{E}_0)$ and

$$|\mathcal{E}[u]| \le r\|u\|^2 + s|\mathcal{E}_0[u]|, \quad u \in \mathrm{Dom}(\mathcal{E}_0), \tag{B.12}$$

where r, s are some non-negative constants. The infimum of all constants s for which a corresponding r exists such that the last inequality holds is called the $\mathcal{E}_0$-bound of $\mathcal{E}$.

A proof of the following result can be found in (Kato 1995), p.320.

Theorem B.4. *Let $\mathcal{E}_0$ be a sectorial form and let $\mathcal{E}$ be $\mathcal{E}_0$-bounded with $\mathcal{E}_0$-bound smaller than* 1. *Then $\mathcal{E}_0 + \mathcal{E}$ is sectorial. Moreover, $\mathcal{E}_0 + \mathcal{E}$ is closed (closable) if and only if $\mathcal{E}_0$ is closed (closable).*

The close connection between sectorial forms and sectorial operators is established via the following representation theorem, see (KATO 1995), p.322.

Theorem B.5. *Let $\mathcal{E}$ be a densely defined, closed, sectorial form in $\mathcal{H}$. Then there exists a unique m-sectorial operator Z such that*

(i) $\mathrm{Dom}(Z) \subset \mathrm{Dom}(\mathcal{E})$ *and* $\mathcal{E}(u,v) = \langle Zu, v\rangle$ *for every* $u \in \mathrm{Dom}(Z)$ *and* $v \in \mathrm{Dom}(\mathcal{E})$.

(ii) $\mathcal{E}$ *is the closure of* $\mathcal{E}|_{\mathrm{Dom}(Z)}$, *i.e.,* $\mathrm{Dom}(Z)$ *is a core of* $\mathcal{E}$.

(iii) If $u \in \mathrm{Dom}(\mathcal{E})$, $w \in \mathcal{H}$ and $\mathcal{E}(u,v) = \langle w, v\rangle$ holds for every v belonging to a core of $\mathcal{E}$, then $u \in \mathrm{Dom}(Z)$ and $Zu = w$.

In particular, if Z is the m-sectorial operator associated with the form $\mathcal{E}$, then the previous theorem implies that the numerical range of Z is a dense subset of the numerical range of $\mathcal{E}$.

We conclude this appendix with an approximation result.

Theorem B.6. *Let $\mathcal{E}$ be a densely defined, closed, sectorial form and let $\mathcal{E}_n$ be a sequence of forms with $\mathrm{Dom}(\mathcal{E}_n) = \mathrm{Dom}(\mathcal{E})$ such that*

$$|(\mathcal{E} - \mathcal{E}_n)[u]| \leq r_n\|u\|^2 + s_n \operatorname{Re}(\mathcal{E})[u], \quad u \in \mathrm{Dom}(\mathcal{E}),$$

where the constants $r_n, s_n > 0$ tend to zero as $n \to \infty$. Then the following holds:

(i) The forms $\mathcal{E}_n$ are closed and sectorial for sufficiently large n.

(ii) If Z and Z_n denote the m-sectorial operators associated to $\mathcal{E}$ and $\mathcal{E}_n$, respectively, then every $\lambda \in \rho(Z)$ belongs to $\rho(Z_n)$ for sufficiently large n, and we have

$$\|R_Z(\lambda) - R_{Z_n}(\lambda)\| \to 0 \tag{B.13}$$

as $n \to \infty$.

For a proof we refer to (KATO 1995), p.339.

C. A non-selfadjoint rank one perturbation: Katriel's example

Let $Z_0 \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator with $\sigma(Z_0) = [a,b]$. If $M \in \mathcal{F}(\mathcal{H})$ is of rank one, then we know from Remark 1.2.9 that the spectrum of $Z = Z_0 + M$ consists of $[a,b]$ and an at most countable sequence of eigenvalues of finite type, which constitutes the discrete spectrum of Z. In particular, if M is selfadjoint then Proposition 4.1.3 implies that $\sigma_d(Z)$ is a finite subset of $\mathbb{R}$ and that Z can have at most one eigenvalue above b and below a, respectively. In this appendix, we will show that for non-selfadjoint M the discrete spectrum of Z need not be finite.

The following proposition (to be found in (Demuth et al. 2008), Section 6) is due to G. Katriel. As above, $J_0 \in \mathcal{B}(l^2(\mathbb{Z}))$ denotes the free Jacobi operator, i.e.,

$$(J_0 u)(k) = u(k-1) + u(k+1), \quad u \in l^2(\mathbb{Z}), \quad k \in \mathbb{Z}.$$

We recall from Chapter 5 that $\sigma(J_0) = [-2, 2]$.

Proposition C.1. *Given any sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C} \setminus [-2, 2]$ which satisfies*

$$\sum_{k\in\mathbb{N}} \frac{\operatorname{dist}(\lambda_k, [-2, 2])}{|\lambda_k + 2|^{1/2}|\lambda_k - 2|^{1/2}} < \infty,$$

there exists a rank one operator M on $l^2(\mathbb{Z})$ such that, setting $J = J_0 + M$, we have $\{\lambda_k\} \subset \sigma_d(J)$.

In particular, since we may choose $\lambda_k = k^{-(1+\delta)}i$ or $\lambda_k = (-1)^k(2 + k^{-2(1+\delta)})$, with $\delta > 0$ arbitrarily small, the following corollaries are direct consequences of the previous proposition.

Corollary C.2. *For any $\gamma < 1$ there exists a rank one operator M such that the eigenvalues of $J = J_0 + M$ satisfy*

$$\sum_{\lambda\in\sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^\gamma}{|\lambda + 2|^{\gamma/2}|\lambda - 2|^{\gamma/2}} = +\infty.$$

Corollary C.3. *For any $\gamma < 1/2$ there exists a rank one operator M such that the eigenvalues of $J = J_0 + M$ on the real line satisfy*

$$\sum_{\lambda\in\sigma_d(J),\lambda<-2} |\lambda + 2|^\gamma = \sum_{\lambda\in\sigma_d(J),\lambda>2} |\lambda - 2|^\gamma = +\infty.$$

Proof of Proposition C.1. The rank one perturbation $M = M(\{\alpha_j\})$ is defined by

$$Mu = \Big[\sum_{j=-\infty}^{\infty} \alpha_j u(j)\Big]\delta_0, \quad u \in l^2(\mathbb{Z}),$$

where $\delta_0(0) = 1$ and $\delta_0(k) = 0$ if $k \neq 0$, and $\alpha_j \in \mathbb{C}$ are to be determined below. For M to be bounded on $l^2(\mathbb{Z})$ we need to assume that

$$\sum_{j=-\infty}^{\infty} |\alpha_j|^2 < \infty. \tag{C.1}$$

In the following, we look for eigenvectors $u_w \in l^2(\mathbb{Z})$ of $J = J_0 + M$ of the form

$$u_w(k) = w^{|k|}, \quad k \in \mathbb{Z},$$

with $w \in \mathbb{D}$. To this end, let us note that for $|k| \geq 1$

$$(Ju_w)(k) = w^{|k|}(w^{-1} + w). \tag{C.2}$$

Moreover,

$$(Ju_w)(0) = 2w + \sum_{j=-\infty}^{\infty} \alpha_j w^{|j|} = \alpha_0 + (\alpha_1 + \alpha_{-1} + 2)w + \sum_{j=2}^{\infty}(\alpha_j + \alpha_{-j})w^j. \tag{C.3}$$

By (C.2) we see that if u_w is an eigenvector then the corresponding eigenvalue is $\lambda = w + w^{-1}$. But then (C.3) implies that a necessary and sufficient condition for u_w to be an eigenvector is that

$$\alpha_0 + (\alpha_1 + \alpha_{-1} + 2)w + \sum_{j=2}^{\infty}(\alpha_j + \alpha_{-j})w^j = \lambda = w + w^{-1},$$

which we can rewrite as $\phi(w) = 0$ where $\phi \in H^2(\mathbb{D})$[10] is defined by

$$\phi(w) = -1 + \alpha_0 w + (\alpha_1 + \alpha_{-1} + 1)w^2 + \sum_{j=3}^{\infty}(\alpha_{j-1} + \alpha_{-j+1})w^j. \tag{C.4}$$

Thus, if w is a zero of ϕ in $\mathbb{D}$, then $\lambda = w + w^{-1}$ is an eigenvalue of J in $\mathbb{C} \setminus [-2, 2]$.

Let $\{\lambda_k\} \subset \mathbb{C} \setminus [-2, 2]$ be any sequence satisfying

$$\sum_{k=1}^{\infty} \frac{\operatorname{dist}(\lambda_k, [-2, 2])}{|\lambda_k^2 - 4|^{1/2}} < \infty. \tag{C.5}$$

In the following, we will select a specific sequence $\{\alpha_j\}$ such that $\{\lambda_k\} \subset \sigma_d(J)$, where $J = J_0 + M(\{\alpha_j\})$ is as defined above. To this end, we define the sequence $\{w_k\} \subset \mathbb{D} \setminus \{0\}$ by

$$\lambda_k = w_k + w_k^{-1}.$$

Using Lemma 3.2.1 it is easy to check that condition (C.5) on λ_k is equivalent to

$$\sum_{k=1}^{\infty}(1 - |w_k|) < \infty. \tag{C.6}$$

As discussed in Remark 2.2.6, (C.6) implies the existence of a function $g \in H^2(\mathbb{D})$ (actually, $g \in H^\infty(\mathbb{D})$) with $\mathcal{Z}(g) = \{w_k\}_{k \in \mathbb{N}}$. We can normalize g so that $g(0) = -1$. Denoting the Taylor expansion of g by

$$g(w) = -1 + \sum_{j=1}^{\infty} \beta_j w^j,$$

and noting that $\{\beta_j\} \in l^2(\mathbb{N})$, we can choose $\alpha_0 = \beta_1$, $\alpha_1 = \beta_2 - 1$, $\alpha_j = \beta_{j+1}$ for $j \geq 2$ and $\alpha_j = 0$ for $j < 0$, so that from (C.4) we obtain $\phi = g$. From the considerations above, this implies that $\lambda_k = w_k + w_k^{-1}$ are eigenvalues of $J = J_0 + M(\{\alpha_j\})$. We have thus proven Proposition C.1. ∎

[10] By definition, $h(z) = \sum_{k=0}^{\infty} b_k z^k \in H(\mathbb{D})$ is in $H^2(\mathbb{D})$ if $\{b_k\}_{k \geq 0} \in l^2(\mathbb{N}_0)$. In particular, using Parseval's identity we see that $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$.

Remark C.4. Proposition C.1 is a special instance of the following more general result: If Z_0 is a closed operator in $\mathcal{H}$ and the operator M is defined by

$$Mf = \langle f, g \rangle g_0, \quad f \in \mathcal{H},$$

where $g, g_0 \in \mathcal{H}$ are fixed, then $\lambda \in \rho(Z_0)$ is an eigenvalue of $Z = Z_0 + M$ if and only if

$$\langle R_{Z_0}(\lambda) g_0, g \rangle = 1.$$

For a proof we refer to (Davies 2007), p.334. $\square$

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List of Symbols

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